

Cover Times in the Discrete Cylinder*

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Abstract

This article proves that, in terms of local times, the properly rescaled and re-centered cover times of finite subsets of the discrete cylinder by simple random walk converge in law to the Gumbel distribution, as the cardinality of the set goes to infinity. As applications we obtain several other results related to covering in the discrete cylinder. Our method is new and involves random interlacements, which were introduced in [22]. To enable the proof we develop a new stronger coupling of simple random walk in the cylinder and random interlacements, which is also of independent interest.

0 Introduction

In this article we prove precise results about the asymptotic distribution of cover times of certain finite subsets of the discrete cylinder, with base a d -dimensional torus for $d \geq 2$, using the theory of *random interlacements*. For families of *finite* graphs the cover time C_V of the whole vertex set V has been extensively studied (see for instance [1–3, 5, 8, 11, 13]). For many families one can show that $\mathbb{E}C_V$ is of order $c|V| \log |V|$, and also that $C_V/(c|V| \log |V|) \rightarrow 1$ in probability as $|V| \rightarrow \infty$ (see Chapter 6 of [3]). For a quite restricted class of families of “especially nice graphs”, one can also prove the finer result that $C_V/(c|V|) - \log |V|$ tends in law to the standard Gumbel distribution (see [11, 13]). In this article we are able to prove the corresponding statement for the cover times of subsets F of the discrete cylinder (seen as an infinite graph): we show that $L_{C_F}/(cN^d) - \log |F|$ tends in law to the Gumbel distribution as $|F| \rightarrow \infty$, provided the sets F are close to the zero level, where L_{C_F} is the *local time* at the zero level of the cylinder when F is covered. As applications we obtain several other results related to covering. To prove the Gumbel distributional limit result we develop an improved coupling of simple random walk in the cylinder and random interlacements, which is also of independent interest.

We now introduce the objects of study and our results more precisely. We denote by $\mathbb{T}_N = (\mathbb{Z}/N\mathbb{Z})^d$ the discrete torus and by $E_N = \mathbb{T}_N \times \mathbb{Z}$ the discrete cylinder for $d \geq 2$. Let P be the canonical law of simple random walk in E_N starting uniformly on the zero level $\mathbb{T}_N \times \{0\}$, and let X_n denote the canonical discrete time process. For any finite set $F \subset E_N$ the cover time C_F of F is the first time X_n has visited every vertex of F :

$$C_F = \inf\{n \geq 0 : F \subset X(0, n)\},$$

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where $X(0, n)$ denotes the set of vertices visited up to time n .

We start by stating the applications of our main result. In Corollary 2.1 we show that if $F_N \subset \mathbb{T}_N \times [-\frac{N}{2}, \frac{N}{2}]$ is a sequence of sets such that $|F_N| \rightarrow \infty$, then under P

$$\frac{C_{F_N}}{(N^d \log |F_N|)^2} \xrightarrow{\text{law}} \zeta\left(\frac{g(0)}{\sqrt{d+1}}\right), \text{ as } N \rightarrow \infty, \quad (0.1)$$

where $\zeta(\tau)$ denotes the first time the local time at zero of a Brownian motion reaches τ and $g(\cdot)$ is the \mathbb{Z}^{d+1} Green function (see (1.7)). In [9, 18] the cover time $C_{\mathbb{T}_N \times \{0\}}$ was studied and found to be of order $N^{2d+o(1)}$. The result (0.1) sharpens this and provides the correct form of the log correction term.

To state our second application we introduce L_n , the local time at zero of the \mathbb{Z} -component of X_n (which we often refer to as “the local time of the zero level”). For any $z \in \mathbb{R}$ let \mathcal{N}_N^z be the point process on $(\mathbb{R}/\mathbb{Z})^d \times \mathbb{R}$ defined by:

$$\mathcal{N}_N^z = \sum_{x \in \mathbb{T}_N \times \{0\}} \delta_{x/N} 1_{\{L_{H_x} > N^d u(z)\}}, \quad (0.2)$$

where $u(z) = g(0)\{\log |\mathbb{T}_N \times \{0\}| + z\}$. In other words \mathcal{N}_N^z counts the vertices of $\mathbb{T}_N \times \{0\}$ that are hit after the local time of the zero level reaches $N^d u(z)$ (it will later become clear that $N^d u(0)$ is the “typical” local time at which covering of the zero level of the cylinder is completed). We call \mathcal{N}_N^z the “point process of vertices covered last”. Let λ be Lebesgue measure on $(\mathbb{R}/\mathbb{Z})^d \times \{0\}$. We show in Corollary 2.2 that

$$\begin{aligned} \mathcal{N}_N^z \text{ converges weakly to a Poisson point process} \\ \text{on } (\mathbb{R}/\mathbb{Z})^d \times \mathbb{R} \text{ of intensity } \exp(-z)\lambda. \end{aligned} \quad (0.3)$$

As a consequence we obtain in Corollary 2.3 that

$$\begin{aligned} \text{the last two vertices of } \mathbb{T}_N \times \{0\} \text{ to be visited by } X_n \\ \text{are "far apart" at typical distance of order } N. \end{aligned} \quad (0.4)$$

The proofs of (0.3) and (0.4) also provide similar results with other subsets of $\mathbb{T}_N \times [-\frac{N}{2}, \frac{N}{2}]$ in place of $\mathbb{T}_N \times \{0\}$ (for example for $\mathbb{T}_N \times [-\frac{N}{2}, \frac{N}{2}]$ itself). The three applications (0.1), (0.3) and (0.4) are consequences of the following main theorem and the coupling (see (0.6) below):

Theorem 0.1 (Convergence to Gumbel). *Let $F_N \subset \mathbb{T}_N \times [-\frac{N}{2}, \frac{N}{2}]$, $N \geq 1$, be a sequence of sets such that $|F_N| \rightarrow \infty$, as $N \rightarrow \infty$. Then under P*

$$\frac{L_{C_{F_N}}}{g(0)N^d} - \log |F_N| \xrightarrow{\text{law}} G, \text{ as } N \rightarrow \infty, \quad (0.5)$$

where G denotes the standard Gumbel distribution (see (1.44)).

As mentioned in the first paragraph the class of finite graphs for which one can obtain a Gumbel distributional limit for the cover time is quite restricted (it includes the complete graph, the star graph (see [3]) and graphs that are “highly symmetric” in the sense of [11], but for example not the graph $\mathbb{T}_N, d \geq 3$). An additional interest of Theorem 0.1

stems from the method we employ in its proof, which relies on random interlacements. It is open whether the method could be used to prove Gumbel distributional limits for the cover times of other graphs; for more on this see Remark 6.11 (1).

Before describing the method in more detail let us briefly discuss the random interlacement model. The model was introduced in [22] and helps to understand the “local picture” left by a simple random walk in e.g. the discrete torus $\mathbb{T}_N, d \geq 3$, (see [24]) or the discrete cylinder $E_N, d \geq 2$, (see [19]) when the walk is run up to times of a suitable scale. The random interlacements consist of a Poisson cloud of doubly infinite trajectories module time-shift in $\mathbb{Z}^d, d \geq 3$, where u multiplies the intensity. The trace of the trajectories in the cloud up to a level u is denoted by $\mathcal{I}^u \subset \mathbb{Z}^d$, so that $(\mathcal{I}^u)_{u \geq 0}$ is an increasing family of random sets. Intuitively speaking, for a value u related to the time up to which the random walk is run, the trace of the random walk in a “local box” in the torus or cylinder in some sense “looks like” \mathcal{I}^u . The previous sentence has further been made precise in the case of the cylinder by means of a coupling in [20, 21]. The first main ingredient in the proof of Theorem 0.1 is a strengthened version of this coupling. To state it we fix an $\varepsilon \in (0, 1)$ and let A be a box of side length $N^{1-\varepsilon}$ with centre at x for some $x \in \mathbb{T}_N \times [-\frac{N}{2}, \frac{N}{2}]$. We further let R_k denote the successive returns to $\mathbb{T}_N \times [-N, N]$ and D_k the successive departures from $\mathbb{T}_N \times (-h_N, h_N)$, where h_N has order $N(\log N)^2$, (see (1.2)). Then the coupling result, see Theorem 4.1, implies that:

For N large enough, and for any $u \geq \frac{1}{\sqrt{N}}, \delta \geq \frac{c_2}{(\log N)^2}$, we can construct a coupling

Q_1 of X . under P , and of joint random interlacements $\mathcal{I}^{u(1-\delta)}, \mathcal{I}^{u(1+\delta)}$ for which

$$Q_1(\mathcal{I}^{u(1-\delta)} \cap A \subset X(0, D_{[uK_N]}) \cap A \subset \mathcal{I}^{u(1+\delta)} \cap A) \geq 1 - cuN^{-3d-1}, \quad (0.6)$$

where K_N essentially equals $\frac{N^d}{(d+1)h_N}$ (see (1.16)). In fact (and importantly for our proof of Theorem 0.1), Theorem 4.1 is stronger than what is stated in (0.6) because it couples the trace of X in several disjoint regions of the cylinder with independent random interlacements, as long as these regions are “far apart”.

An interest of (0.6) is that it couples $X(0, D_{[uK_N]})$ with *joint* random interlacements $\mathcal{I}^{u(1-\delta)}$ and $\mathcal{I}^{u(1+\delta)}$ (combining the one-sided couplings of [20, 21] to get a two-sided coupling does not guarantee the correct joint law of $\mathcal{I}^{u(1-\delta)}$ and $\mathcal{I}^{u(1+\delta)}$). This makes it more useful as a “transfer mechanism” from random interlacements to random walk; see Remark 6.11 (2) for more on this topic.

Thanks to the Poissonian structure of random interlacements one has a number of algebraic properties that only hold approximately for the trace of random walk (cf. (1.35), (1.36), (1.37)). In [4] we could take advantage of this feature and give a precise result for the asymptotic distributions of so called *cover levels* in random interlacements. The cover level a of a finite set $F \subset \mathbb{Z}^{d+1}$ is:

$$\tilde{C}_F = \inf\{u \geq 0 : F \subset \mathcal{I}^u\}. \quad (0.7)$$

Theorem 0.1 of [4] implies that, in the notation of (0.5):

$$\frac{\tilde{C}_F}{g(0)} - \log |F| \xrightarrow{\text{law}} G, \text{ as } |F| \rightarrow \infty. \quad (0.8)$$

The method used to prove Theorem 0.1 is essentially speaking to combine (0.8) with the coupling (0.6). It will turn out that when the local time L_n of the zero level is uN^d

then, roughly speaking, there have been about $[uK_N]$ excursions (see (1.23)). On the other hand (0.6) intuitively says that after $[uK_N]$ excursions the picture left in a local box (meaning a box of side length $N^{1-\varepsilon}, \varepsilon > 0$) looks like random interlacements at level u . Thus “when the local time at the zero level is uN^d the picture in a local box looks like \mathcal{I}^u ” (and this also holds simultaneously for the picture left in several “distant” regions contained in local boxes). Now (0.8) essentially speaking says that \tilde{C}_F is close in distribution to $g(0)\{\log|F|+G\}$, and thus we roughly find that if F is contained in one or several “distant” local boxes then L_{C_F} , the local time at the zero level when F is covered, is close in distribution to $N^d g(0)\{\log|F|+G\}$. But this is the intuitive meaning of (0.5). When F is contained in one or several “distant” local boxes this intuitive explanation can be turned into a rigorous proof.

However sets like $F_N = \mathbb{T}_N \times \{0\}$ can not be split into pieces that are contained in distant local boxes. To deal with this problem we consider two cases. The first, considered in Proposition 3.1, is when the F_N are small in the sense that $|F_N| \leq N^{1/8}$. It turns out that we can split such small sets into pieces S_1, S_2, \dots, S_k such that the pieces are contained in “distant” local boxes, so that we are in the situation discussed in the previous paragraph and can prove that the limit distribution is the Gumbel distribution.

The second case, considered in Proposition 3.2, is when the sets are “large” in the sense that $|F_N| > N^{1/8}$. It turns out that such a set is typically covered completely when the *local* time (at the zero level) reaches roughly $N^d g(0) \log|F_N|$. We consider the set F_N^ρ of vertices not covered when the local time at the zero level reaches a fraction $(1-\rho)$ of the typical local time $N^d g(0) \log|F_N|$ (in fact F_N^ρ will be defined in terms of excursions). By tiling the cylinder with local boxes, using the coupling (0.6) once for each box, and using a calculation inside the random interlacements model we are able to show (for appropriate values of ρ) that F_N^ρ is with high probability “small” in the sense that $|F_N^\rho| \leq N^{1/8}$ and that $|F_N^\rho|$ concentrates around its typical value, which turns out to be $|F_N|^\rho$. By excluding a short segment of the random walk (when it is far away from F_N and thus does not affect F_N^ρ) during which it “forgets” the shape of F_N^ρ we will show that the way in which X covers F_N^ρ is essentially the same as the way an independent random walk would cover F_N^ρ . Thus $L_{C_{F_N}}$ should be close in distribution to $(1-\rho)N^d g(0) \log|F_N| + L_{C_{F'}}$, where F' is independent from X and distributed as F_N^ρ . Since F_N^ρ is “small” with high probability we can apply the previous case for typical realisations of F' to get that $L_{C_{F'}}$ is close in distribution to $N^d g(0)\{\log|F'|+G\}$. Since $|F_N^\rho|$ concentrates around $|F_N|^\rho$ we find that $\log|F'| \approx \rho \log|F_N|$ so adding $L_{C_{F'}}$ to the deterministic part $(1-\rho)N^d g(0) \log|F_N|$ we get that $L_{C_{F_N}}$ has law close to $N^d g(0)\{\log|F_N|+G\}$ (which is the intuitive interpretation of (0.5)).

We now describe how this article is organized. In Section 1 we fix notation, recall some standard results on random walks and random interlacements and prove some preliminary lemmas. In Section 2 we use our main result Theorem 0.1 to prove (0.1), (0.3) and (0.4). In Section 3 we then prove Theorem 0.1, using the full version of the coupling (0.6) (i.e. Theorem 4.1), and a quantitative version of (0.8) (see (1.45)). The proof of Theorem 4.1 is contained in sections 4, 5 and 6.

Finally a note on constants. Named constants are denoted by c_0, c_1, \dots and have fixed values. Unnamed constants are denoted by c and may change from line to line and within formulas. All constants are strictly positive and unless otherwise indicated they only

depend on d . Further dependence on e.g. parameters α, β is denoted by $c(\alpha, \beta)$.

1 Notation and some useful results

In this section we fix notation and recall some known results about random walk and random interacements. We also state and prove Lemma 1.2 which gives an upper bound on a certain killed Green function in the cylinder, Lemma 1.4 which relates local time of the random walk to excursion times and to Brownian local time, and Lemma 1.5 which gives a bound on certain sums of the “two point function” in the random interacements model.

In this article $\mathbb{N} = \{0, 1, 2, \dots\}$. For any real $x \geq 0$ we denote the integer part of x by $[x]$. If U is a set $|U|$ denotes the cardinality of U .

We denote by $|\cdot|_\infty$ and $|\cdot|$ the l_∞ and Euclidean norms on \mathbb{R}^{d+1} and by $d_\infty(\cdot, \cdot)$ and $d(\cdot, \cdot)$ the corresponding induced distances on $(\mathbb{R}/\mathbb{Z})^d \times \mathbb{R}$, \mathbb{Z}^{d+1} , and E_N . For any two sets $A, B \subset \mathbb{Z}^{d+1}$ or $A, B \subset E_N$ we denote their mutual Euclidean distance $\inf_{x \in A, y \in B} d(x, y)$ by $d(A, B)$. The closed l_∞ -ball centred at x in \mathbb{Z}^{d+1} or E_N of radius R is denoted by $B(x, R)$. For any set $U \subset \mathbb{Z}^{d+1}$ or E_N we define the inner and outer boundaries by

$$\partial_i U = \{x \in U : d(\{x\}, U^c) = 1\} \text{ and } \partial_e U = \{x \in U^c : d(\{x\}, U) = 1\}.$$

A trajectory (or path) is a sequence $w(n), n \in \mathbb{N}$, in \mathbb{Z}^{d+1} or E_N such that $d(w(n+1), w(n)) \leq 1$ for all $n \geq 0$. We define the trace of the trajectory as follows:

$$w(a, b) = \{x : w(n) = x \text{ for some } n \in [a, b]\}, a \leq b \text{ in } \mathbb{N}. \quad (1.1)$$

We write \mathcal{T} for the space of trajectories in E_N and W for the space of trajectories in \mathbb{Z}^{d+1} . For any set $F \subset E_N$ or $F \subset \mathbb{Z}^{d+1}$ we write \mathcal{T}_F for the countable subset of \mathcal{T} consisting of trajectories that are contained in $F \cup \partial F$ and stay constant after a finite time. The canonical coordinates on \mathcal{T} and W are denoted by $(X_n)_{n \geq 0}$ and the canonical shift by $(\theta_n)_{n \geq 0}$. For a subset U of E_N or \mathbb{Z}^{d+1} we define the entrance time H_U , the hitting time \tilde{H}_U , and the exit time T_U by:

$$\begin{aligned} H_U &= \inf\{n \geq 0 : X_n \in U\}, \tilde{H}_U = \inf\{n \geq 1 : X_n \in U\}, \\ T_U &= \inf\{n \geq 0 : X_n \notin U\}. \end{aligned}$$

When U is the singleton $\{x\}$ we write H_x or \tilde{H}_x for simplicity. We define the special levels r_N, h_N and the special slabs B, \tilde{B} of E_N by

$$r_N = N, h_N = [N(2 + (\log N)^2)] \text{ and } B = \mathbb{T}_N \times [-r_N, r_N], \tilde{B} = \mathbb{T}_N \times (-h_N, h_N). \quad (1.2)$$

The successive returns to B and departures from \tilde{B} are given by

$$\begin{aligned} R_1 &= H_B, D_1 = T_{\tilde{B}} \circ \theta_{R_1} + R_1, \text{ and for } k \geq 1, \\ R_{k+1} &= R_1 \circ \theta_{D_k} + D_k, \text{ and } D_k = D_1 \circ \theta_{D_k} + D_k. \end{aligned} \quad (1.3)$$

For $x \in \mathbb{Z}^{d+1}$ we denote by $P_x^{\mathbb{Z}^{d+1}}$ the law on W of simple random starting at x . For $x \in E_N$ we denote by P_x the law on \mathcal{T} of simple random starting at x . If e is a measure on \mathbb{Z}^{d+1}

or E_N we denote by $P_e^{\mathbb{Z}^{d+1}}$, P_e the measures $\sum_a e(a)P_e^{\mathbb{Z}^{d+1}}$ and $\sum_a e(a)P_e$ respectively. A special role will be played by the measures

$$q = \frac{1}{2N^d} \sum_{x \in \mathbb{T}_N \times \{-r_N, r_N\}} \delta_x \text{ and } q_z = \frac{1}{N^d} \sum_{x \in \mathbb{T}_N \times \{z\}} \delta_x, z \in \mathbb{Z}. \quad (1.4)$$

Note that the measure P that appears in the introduction coincides with P_{q_0} . For any finite $K \subset \mathbb{Z}^{d+1}$ we define the escape probability (or equilibrium measure) e_K and capacity $\text{cap}(K)$ by

$$e_K(x) = P_x^{\mathbb{Z}^{d+1}}(\tilde{H}_K = \infty)1_K(x) \text{ and } \text{cap}(K) = \sum_{x \in K} e_K(x). \quad (1.5)$$

If $K \subset U \subset E_N$ with U finite, then we define the escape probability and capacity of K relative to U by

$$e_{K,U}(x) = P_x(\tilde{H}_K > T_U)1_K(x) \text{ and } \text{cap}_U(K) = \sum_{x \in K} e_{K,U}(x). \quad (1.6)$$

We define the \mathbb{Z}^{d+1} Green function by

$$g(x, y) = \sum_{n \geq 0} P_x^{\mathbb{Z}^{d+1}}(X_n = y) \text{ and } g(\cdot) = g(\cdot, 0) \text{ for } x, y \in \mathbb{Z}^{d+1}. \quad (1.7)$$

The Green function killed on exiting U for $U \subset \mathbb{Z}^{d+1}$ is defined by

$$g_U(x, y) = \sum_{n \geq 0} P_x^{\mathbb{Z}^{d+1}}(X_n = y, n < T_U),$$

and similarly for $U \subset E_N$ with P_x in place of $P_x^{\mathbb{Z}^{d+1}}$. Classically, if $K \subset U \subset E_N$ with U finite, then for all $x \in U$

$$P_x(H_K < T_U) = \sum_{y \in K} g_U(x, y)e_{K,U}(y). \quad (1.8)$$

For two disjoint sets $S_1, S_2 \subset \tilde{B}$ we define their “mutual energy” relative to \tilde{B} :

$$\mathcal{E}(S_1, S_2) = \sum_{x \in S_1, y \in S_2} e_{S_1, \tilde{B}}(x)g_{\tilde{B}}(x, y)e_{S_2, \tilde{B}}(y). \quad (1.9)$$

The following classical bounds on the Green function $g(x)$ follow Theorem 1.5.4 p. 31 of [12]:

Lemma 1.1. *($d \geq 2$) For all non-zero $x \in \mathbb{Z}^{d+1}$*

$$c|x|^{1-d} \leq g(x) \leq c|x|^{1-d}. \quad (1.10)$$

We also have similar bounds on $g_{\tilde{B}}(x, y)$:

Lemma 1.2. ($d \geq 2, N \geq 1$) For any $x, y \in B$ with $x \neq y$

$$c|x - y|^{1-d} \leq g_{\tilde{B}}(x, y) \leq c|x - y|^{1-d} + c\frac{h_N}{N^d}. \quad (1.11)$$

Proof. Let e denote the vector $(0, \dots, 0, 1) \in \mathbb{Z}^{d+1}$. By “unwrapping” the cylinder E_N we see that for any $x, y \in \tilde{B}$

$$g_{\tilde{B}}(x, y) = \sum_{n \in \mathbb{Z}^{d+1}, n \cdot e = 0} g_U(x', y' + nN), \quad (1.12)$$

where $U = \{x \in \mathbb{Z}^{d+1} : |x \cdot e| < h_N\}$ and $x', y' \in \mathbb{Z}^d \times [-r_N, r_N]$ are representatives in \mathbb{Z}^{d+1} of x, y such that $|x - y| = |x' - y'|$. Now the lower bound of (1.11) is a consequence of $g_{\tilde{B}}(x, y) \geq g_U(x', y') \geq g_{B(y', \frac{1}{3}h_N)}(x', y') \geq c|x' - y'|^{1-d}$ where the last inequality follows from Proposition 1.5.9 p. 35 of [12]. Furthermore it follows from (2.13) of [17] with $L = h_N$ that if $n \cdot e = 0$

$$g_U(x', y' + nN) \leq c|x - y|^{1-d} 1_{\{|n| < 3\}} + \frac{1}{(|n|N)^{d-1}} \exp\left(-c\frac{N|n|}{h_N}\right) 1_{\{|n| \geq 3\}}.$$

But $\sum_{n \in \mathbb{Z}^d} \frac{1}{|n|^{d-1}} \exp(-c\frac{N|n|}{h_N}) \leq c\frac{h_N}{N}$, so summing over n in (1.12) one obtains the upper bound of (1.11). \square

Note that thanks to (1.11) we have the following bound on $\mathcal{E}(S_1, S_2)$ when $S_1, S_2 \subset B$:

$$\begin{aligned} \mathcal{E}(S_1, S_2) &\stackrel{(1.6), (1.11)}{\leq} c\text{cap}_{\tilde{B}}(S_1)\text{cap}_{\tilde{B}}(S_2) \left\{ (d(S_1, S_2))^{1-d} + \frac{h_N}{N^d} \right\} \\ &\stackrel{(1.6)}{\leq} c|S_1||S_2| \left\{ (d(S_1, S_2))^{1-d} + \frac{h_N}{N^d} \right\}. \end{aligned} \quad (1.13)$$

The equalities contained in the following lemma will be essential:

Lemma 1.3. ($N \geq 3$) For all $K \subset \mathbb{T}_N \times (-r_N, r_N)$

$$P_q(H_K < T_{\tilde{B}}, X_{H_K} = x) = \frac{1}{K_N} e_{K, \tilde{B}}(x), x \in K \text{ and} \quad (1.14)$$

$$P_q(H_K < T_{\tilde{B}}, (X_{H_K+}) \in dw) = \frac{1}{K_N} P_{e_{K, \tilde{B}}}(dw), \quad (1.15)$$

where

$$K_N = \frac{N^d}{(d+1)(h_N - r_N)}. \quad (1.16)$$

Proof. (1.14) follows from Lemma 1.1 of [20] and (1.15) follows from (1.14) by an application of the strong Markov property. \square

Incidentally (1.14) can be used to see that $\text{cap}_{\tilde{B}}(\{x\}) \leq \text{cap}_{\tilde{B}}(K)$ when $x \in K \subset \mathbb{T}_N \times (-r_N, r_N)$ and therefore together with the bound $\text{cap}_{\tilde{B}}(\{x\}) \geq P_x^{\mathbb{Z}^{d+1}}(T_{B(y, \frac{1}{4}N)} > \tilde{H}_x) - \sup_{y \in \partial B(y, \frac{1}{4}N)} g_{\tilde{B}}(x, y) \stackrel{(1.11)}{\geq} c$ valid for all $x \in \mathbb{T}_N \times [r_N, r_N]$ and $N \geq c$ we see that

$$\mathcal{E}(S_1, S_2) \stackrel{(1.9), (1.11)}{\geq} cN^{1-d} \text{ for all } N \geq 1 \text{ and non-empty } S_1, S_2 \subset \mathbb{T}_N \times (-r_N, r_N). \quad (1.17)$$

The local time of X_n at the zero level (or equivalently the local time at 0 of the \mathbb{Z} -component of X_n) is denoted by

$$L_n = |\{i \in [0, n] : X_i \in \mathbb{T}_N \times \{0\}\}|, n \in \mathbb{N}, \quad (1.18)$$

and the first time the local time at the zero level is at least u by

$$\gamma_u = \inf\{n \geq 0 : L_n \geq u\}, u \geq 0. \quad (1.19)$$

Similarly we define

$$\zeta(u) = \inf\{t \geq 0 : \hat{L}_t \geq u\}, \quad (1.20)$$

where the continuous process \hat{L}_t is the local time at zero of a canonical Brownian motion. Note that by the scaling invariance of Brownian motion

$$\zeta(u) \text{ satisfies the scaling relation } \zeta(u) \stackrel{\text{law}}{=} u^2 \zeta(1), u > 0. \quad (1.21)$$

The cumulative distribution function of $\zeta(u)$ is known explicitly (see e.g. Theorem 2.3 p. 240 of [15]) it is the continuous function

$$F(z) = 1_{\{z > 0\}} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{-\frac{u}{\sqrt{z}}} e^{-x^2/2} dx. \quad (1.22)$$

The following lemma relates γ_u to the excursion times D_k and to the law of $\zeta(\cdot)$:

Lemma 1.4. *For any $N \geq c$, $\frac{1}{2} > \delta \geq c_0 \frac{r_N}{h_N}$ and u such that $uK_N \geq 2$ we have*

$$P(D_{[(1-\delta)uK_N]} < \gamma_{uN^d} \leq D_{[(1+\delta)uK_N]}) \geq 1 - c \exp(-cu\sqrt{N}). \quad (1.23)$$

Also under P , for any fixed $N \geq 3$,

$$\frac{\gamma_u}{u^2} \xrightarrow{\text{law}} \zeta\left(\frac{1}{\sqrt{d+1}}\right), \text{ as } u \rightarrow \infty. \quad (1.24)$$

Proof. We start with (1.23). Note that

$$\{\gamma_a \leq b\} \stackrel{(1.18), (1.19)}{=} \{a \leq L_{[b]}\} \text{ for real } a, b \geq 0. \quad (1.25)$$

Thus it suffices to show (using also that $\delta^2 K_N \geq c\sqrt{N}$)

$$P(L_{D_{[(1-\delta)uK_N]}} < uN^d \leq L_{D_{[(1+\delta)uK_N]}}) \geq 1 - c \exp(-c\delta^2 uK_N). \quad (1.26)$$

Define the successive returns $\tilde{R}_k, k \geq 1$, to $\mathbb{T}_N \times \{0\}$ and departures $\tilde{D}_k, k \geq 1$, from $\mathbb{T}_N \times \{0\}$ analogously to (1.3) with $\mathbb{T}_N \times \{0\}$ replacing both B and \tilde{B} . Let $V_n = |\{k : \tilde{D}_k \in [R_n, D_n]\}|$ denote the number of contiguous intervals of time spent in the zero level during the n -th excursion between B and $\partial_e \tilde{B}$. Then

$$L_{D_n} = \sum_{k=1}^{V_1 + \dots + V_n} (\tilde{D}_k - \tilde{R}_k) \text{ for all } n \geq 1. \quad (1.27)$$

By the strong Markov property $V_n, n \geq 1$, are independent and V_1 is geometric with support $\{1, 2, \dots\}$ and parameter $\frac{1}{h_N}$ (the probability $P_x(\tilde{R}_2 > D_1)$ when $x \in \mathbb{T}_N \times \{0\}$) and $V_n \stackrel{\text{law}}{=} UV_1$ for $n \geq 2$, where U is a Bernoulli random variable, independent from V_1 , with $\mathbb{P}(U = 0) = \frac{r_N}{h_N}$ (the probability that X_n leaves \tilde{B} before hitting $\mathbb{T}_N \times \{0\}$, when starting in $\partial_i B$). By a standard large deviation bound using the exponential Chebyshev inequality and the small exponential moments of $\frac{V_i}{h_N}$ we see that if $\delta \geq 4\frac{r_N}{h_N}$ then

$$\begin{aligned} P(V_1 + \dots + V_{[(1+\delta)uK_N]} &\leq (1 + \frac{\delta}{2})uK_N h_N) \leq \exp(-c\delta^2 uK_N) \text{ and} \\ P(V_1 + \dots + V_{[(1-\delta)uK_N]} &\geq (1 - \frac{\delta}{2})uK_N h_N) \leq \exp(-c\delta^2 uK_N). \end{aligned}$$

Combining this with (1.27) we see that (1.26) (and therefore also (1.23)) follows once we show that

$$P\left(\sum_{k=1}^{[(1-\frac{\delta}{2})uK_N h_N]} (\tilde{D}_k - \tilde{R}_k) < uN^d \leq \sum_{k=1}^{[(1+\frac{\delta}{2})uK_N h_N]} (\tilde{D}_k - \tilde{R}_k)\right) \geq 1 - c \exp(-c\delta^2 uK_N). \quad (1.28)$$

Now by the strong Markov property $\tilde{D}_k - \tilde{R}_k, k \geq 1$, are independent geometric random variables with support $\{1, 2, \dots\}$ and parameter $\frac{1}{d+1}$ (the probability that $X_{n+1} \notin \mathbb{T}_N \times \{0\}$ conditioned on $X_n \in \mathbb{T}_N \times \{0\}$). Therefore by a standard large deviation bound we see that

$$\begin{aligned} P\left(\sum_{k=1}^{[(1-\frac{\delta}{2})uK_N h_N]} (\tilde{D}_k - \tilde{R}_k) \geq (1 - \frac{\delta}{4})(d+1)uK_N h_N\right) &\leq c \exp(-c\delta^2 uK_N h_N) \text{ and} \\ P\left(\sum_{k=1}^{[(1+\frac{\delta}{2})uK_N h_N]} (\tilde{D}_k - \tilde{R}_k) \leq (d+1)uK_N h_N\right) &\leq c \exp(-c\delta^2 uK_N h_N). \end{aligned}$$

From this (1.28) follows by observing that $(1 - \frac{\delta}{4})(d+1)uK_N h_N \stackrel{(1.16)}{=} uN^d \frac{1-\delta/4}{1-r_N/h_N} \leq uN^d \stackrel{(1.16)}{\leq} (d+1)uK_N h_N$ if $\delta \geq 4\frac{r_N}{h_N}$. This completes the proof of (1.23).

We now turn to (1.24). For fixed $N \geq 3$ let Z_n denote the \mathbb{Z} -component of X_n . Then L_n is the local time of Z_n at zero. By (1.22) of [6] we can couple L with \hat{L} , the local time at 0 of a Brownian motion, so that

$$\sup_{n \geq 1} n^{-3/8} |(d+1)\hat{L}_{\frac{n}{d+1}} - L_n| < \infty \text{ a.s.} \quad (1.29)$$

For any $u \geq 0, z \geq 0$ we have $P(\gamma_u \leq zu^2) = P(u \leq L_{[zu^2]})$ by (1.25) and thus it follows from (1.29) that for any $\alpha \in (0, 1)$

$$\begin{aligned} \lim_{u \rightarrow \infty} P(u(1+\alpha) \leq (d+1)\hat{L}_{\frac{[zu^2]}{d+1}}) &\leq \overline{\lim}_{u \rightarrow \infty} P(\gamma_u \leq zu^2) \\ &\leq \overline{\lim}_{u \rightarrow \infty} P(u(1-\alpha) \leq (d+1)\hat{L}_{\frac{[zu^2]}{d+1}}). \end{aligned} \quad (1.30)$$

But by (1.20) and (1.21) we have $P(u(1 \pm \alpha) \leq (d+1)\hat{L}_{\frac{[zu^2]}{d+1}}) = P(\zeta(\frac{1}{\sqrt{d+1}}) \leq \frac{[zu^2]}{u^2(1 \pm \alpha)^2})$ and therefore from (1.30)

$$\mathbb{P}(\zeta(\frac{1}{\sqrt{d+1}}) \leq \frac{z}{(1+\alpha)^2}) \stackrel{(1.22)}{\leq} \varlimsup_{u \rightarrow \infty} P(\gamma_u \leq zu^2) \stackrel{(1.22)}{\leq} \mathbb{P}(\zeta(\frac{1}{\sqrt{d+1}}) \leq \frac{z}{(1-\alpha)^2}).$$

Thus taking $\alpha \rightarrow 0$ we get (1.24). \square

In the proof of the coupling result (0.6) (i.e. Theorem 4.1) the first step is to couple random walk with so called ‘‘Poisson Processes of Excursions’’. To introduce them we first define for any law e on E_N the probability

$$\kappa_e(dw) = P_e(X_{\cdot \wedge T_{\bar{B}}} \in dw). \quad (1.31)$$

A special role will be played by κ_q where q as in (1.4). A ‘‘Poisson process of excursions’’ is a Poisson process on the space $\mathcal{T}_{\bar{B}}$ (see below (1.1)) of intensity which is a multiple of

$$\nu(dw) = K_N \kappa_q(dw) = K_N P_q(X_{\cdot \wedge T_{\bar{B}}} \in dw). \quad (1.32)$$

If $\mu = \sum_{n \geq 0} \delta_{w_n}$ is a point process on one of the spaces $\mathcal{T}_{\bar{B}}$ or W we define the trace $\mathcal{I}(\mu)$ of μ by (see (1.1) for notation)

$$\mathcal{I}(\mu) = \bigcup_{n \geq 0} w_n(0, \infty) \subset E_N \text{ or } \mathbb{Z}^{d+1}. \quad (1.33)$$

We now recall some facts about random interlacements. They are defined as a Poisson point process on a certain space of trajectories modulo time-shift, on a probability space we denote by $(\Omega_0, \mathcal{A}_0, Q_0)$. For a detailed construction we refer to Section 1 of [22] or Section 1 of [16]. In this article we will only need the facts that now follow. On $(\Omega_0, \mathcal{A}_0, Q_0)$ there is a family $(\mathcal{I}^u)_{u \geq 0}$ of random subsets of \mathbb{Z}^{d+1} , indexed by a parameter u . We call \mathcal{I}^u , or any random set with the law of \mathcal{I}^u , a *random interlacement at level u* . Intuitively speaking \mathcal{I}^u is the trace of the Poisson cloud of trajectories mentioned in the introduction. Two basic properties of random interlacements are that

$$\begin{aligned} &\text{the } \mathcal{I}^u \text{ are translation invariant and increasing,} \\ &\text{in the sense that if } v \leq u \text{ then } \mathcal{I}^v \subset \mathcal{I}^u. \end{aligned} \quad (1.34)$$

We can characterise the law of $\mathcal{I}^u \cap K$ for finite sets $K \subset \mathbb{Z}^{d+1}$ in the following manner (see (1.18), (1.20), (1.53) of [22])

$$\begin{aligned} \mathcal{I}^u \cap K &\stackrel{\text{law}}{=} \mathcal{I}(\mu_{K,u}) \cap K \text{ for each } u \geq 0 \text{ where } \mu_{K,u} \text{ is a} \\ &\text{Poisson point process on } W \text{ of intensity } uP_{e_K}^{\mathbb{Z}^{d+1}}. \end{aligned} \quad (1.35)$$

An important fact is that if $u \leq v$ and \mathcal{I}_1 and \mathcal{I}_2 are independent random interlacements then

$$(\mathcal{I}_1^u, \mathcal{I}_1^v) \stackrel{\text{law}}{=} (\mathcal{I}_1^u, \mathcal{I}_1^u \cup \mathcal{I}_2^{v-u}). \quad (1.36)$$

The law of $(\mathcal{I}^u)^c$ (also called the vacant set) on $\{0, 1\}^{\mathbb{Z}^{d+1}}$ is characterized by (see (2.16) of [22]):

$$Q_0(A \subset (\mathcal{I}^u)^c) = \exp(-u \cdot \text{cap}(A)) \text{ for all finite } A \subset \mathbb{Z}^{d+1}.$$

Since $\text{cap}(\{x\}) = \frac{1}{g(0)}$ and $\text{cap}(\{x, y\}) = \frac{2}{g(x)+g(x-y)}$ (see (1.62) and (1.64) of [22]) we have:

$$Q_0(x \notin \mathcal{I}^u) = \exp\left(-\frac{u}{g(0)}\right) \text{ and } Q_0(x, y \notin \mathcal{I}^u) = \exp\left(-u \frac{2}{g(0) + g(x-y)}\right). \quad (1.37)$$

If A and B are two disjoint finite sets in \mathbb{Z}^{d+1} that are “far apart” we have the following independence result which is a direct consequence of Lemma 2.1 of [4]

$$|Q_0(A \subset \mathcal{I}^u, B \subset \mathcal{I}^u) - Q_0(A \subset \mathcal{I}^u)Q_0(B \subset \mathcal{I}^u)| \leq cu \frac{\text{cap}(A)\text{cap}(B)}{(d(A, B))^{d-1}} \quad (1.38)$$

for all disjoint $A, B \subset \mathbb{Z}^{d+1}$ and $u > 0$.

The next lemma gives a bound on certain sums of the “two point probability” $Q_0(x, y \notin \mathcal{I}^u)$. It is a generalisation of Lemma 2.5 of [4].

Lemma 1.5. *($d \geq 2$) There is a constant $c_1 > 1$ such that for any $F \subset \mathbb{Z}^{d+1}$, $u \geq 0$ and $a > 0$*

$$\sum_{0 < |x| < a, x \in F} Q_0(0, x \notin \mathcal{I}^{g(0)u}) \leq e^{-2u} \{|F| \wedge a^{d+1} + cu|F|^{\frac{2}{d+1}}\} + ce^{-c_1 u}. \quad (1.39)$$

Proof. We assume $|F| \geq 1$. The left-hand side of (1.39) equals $I_1 + I_2$ where

$$I_1 \stackrel{(1.37)}{=} \sum_{0 < |x| < u \wedge a, x \in F} \exp\left(-2u(1 + g(x)/g(0))^{-1}\right) \text{ and} \quad (1.40)$$

$$I_2 \stackrel{(1.37)}{=} \sum_{u \wedge a \leq |x| < a, x \in F} \exp\left(-2u(1 + g(x)/g(0))^{-1}\right). \quad (1.41)$$

We first bound I_1 . Note that for $x \neq 0$ we have $g(x) < g(e_1) < g(0)$, where e_1 is a unit vector in \mathbb{Z}^d , so that the summand in (1.40) is bounded by $\exp(-c'_1 u)$ where $c'_1 = 2(1 + g(1)/g(0))^{-1} > 1$. Therefore we find that

$$I_1 \leq cu^{d+1} \exp(-c'_1 u) \leq c \exp(-c_1 u) \quad (1.42)$$

for a constant c_1 such that $c'_1 > c_1 > 1$. Next to bound I_2 we use the elementary inequality $(1 - x)^{-1} \geq 1 + x, x \geq 0$ to get

$$\begin{aligned} I_2 &\leq \sum_{u \wedge a \leq |x| < a, x \in F} \exp(-2u(1 - g(x)/g(0))) \\ &\leq \exp(-2u) \sum_{u \wedge a \leq |x| < a, x \in F} (1 + cug(x)/g(0)) \end{aligned}$$

where in the last inequality we have used that $ug(x) \stackrel{(1.10)}{\leq} (cu \cdot u^{1-d} \wedge c) \leq c$ for $u \wedge a \leq |x| < a$. We thus get

$$I_2 \leq \exp(-2u) \left\{ |F| \wedge (ca^{d+1}) + cu \sum_{x \in F} g(x) \right\}.$$

Let $F = \{f_1, \dots, f_{|F|}\}$ with $|f_1| \leq |f_2| \leq \dots \leq |f_{|F|}|$ and let h_1, h_2, \dots be an enumeration of \mathbb{Z}^{d+1} such that $|h_1| \leq |h_2| \leq \dots$. Then since $g(f_i) \stackrel{(1.10)}{\leq} c|f_i|^{1-d} \leq c|h_i|^{1-d}$ and $\{h_1, \dots, h_{|F|}\}$ is contained in the Euclidean ball B' of radius $c|F|^{\frac{1}{d+1}}$ around the origin we have $\sum_{x \in F} g(x) \leq \sum_{i=1}^{|F|} (c|h_i|^{1-d} \wedge g(0)) \leq \sum_{x \in B'} (c|x|^{1-d} \wedge g(0)) \leq c|F|^{2/(d+1)}$. Therefore we see that

$$I_2 \leq c \exp(-2u) \{ |F| \wedge a^{d+1} + cu|F|^{2/(d+1)} \},$$

which in combination with (1.42) yields (1.39). \square

In proving Theorem 0.1 we will often consider events similar to $\{L_{C_{F_N}} \leq N^d g(0) \{\log |F_N| + z\}\}$. To simplify formulas we define

$$u_F(z) = g(0) \{\log |F| + z\} \text{ and } u_N = u_{F_N}, \quad (1.43)$$

so that the previous event coincides with $\{L_{C_{F_N}} \leq N^d u_N(z)\}$. We denote the cumulative distribution function of the standard Gumbel distribution by

$$G(z) = \exp(-e^{-z}). \quad (1.44)$$

We now state a quantitative version of (0.8) which will be what we actually use in the proof of Theorem 0.1. Recall the definition (0.7) of the cover level \tilde{C}_F of a set F . We have that (see Theorem 0.1 of [4]) for all finite non-empty $F \subset \mathbb{Z}^{d+1}$

$$\sup_{z \geq -\log |F|} |Q_0(\tilde{C}_F \leq u_F(z)) - G(z)| \leq c|F|^{-c}. \quad (1.45)$$

2 Applications: Convergence of cover times and point process of uncovered vertices

In this section we will derive from Theorem 0.1 and the coupling (0.6) the convergence in law of the rescaled cover times (i.e. (0.1)), the convergence in law of the point process of vertices covered last (i.e. (0.3)) and the statement that “the last two vertices to be hit are far apart” (i.e. (0.4)), in the form of Corollary 2.1, Corollary 2.2 and Corollary 2.3 respectively. Corollary 2.1 will follow easily from Theorem 0.1 once we have related the random walk local time L_n to the local time of Brownian motion using (1.24). To prove Corollary 2.2 we will use Kallenberg’s theorem which allows us to conclude that \mathcal{N}_N^z converges weakly to a homogeneous Poisson point process by checking two conditions involving convergence of the intensity measure and the probability that the point measure does not charge a set. Finally Corollary 2.3 follows from Corollary 2.2 by a calculation involving the Palm measure of the limit of the \mathcal{N}_N^z .

Corollary 2.1 (Convergence in law for cover times). *($d \geq 2$) Let F_N be a sequence of sets as in Theorem 0.1. Then under P*

$$\frac{C_N}{(N^d \log |F_N|)^2} \xrightarrow{\text{law}} \zeta\left(\frac{g(0)}{\sqrt{d+1}}\right), \text{ as } N \rightarrow \infty, \quad (2.1)$$

where $C_N = C_{F_N}$.

Proof. We set $u = N^d \log |F_N|$. For any $z \geq 0$ we have

$$\{L_{C_N} < L_{[zu^2]}\} \stackrel{(1.18)}{\subset} \{C_N \leq zu^2\} \stackrel{(1.18)}{\subset} \{L_{C_N} \leq L_{[zu^2]}\}. \quad (2.2)$$

It follows from (0.5) that $L_{C_N}/u \rightarrow g(0)$ in probability as $N \rightarrow \infty$, so that for any $\delta \in (0, 1)$

$$\begin{aligned} \lim_{N \rightarrow \infty} P(g(0)(1 + \delta)u \leq L_{[zu^2]}) &\stackrel{(2.2)}{\leq} \overline{\lim}_{N \rightarrow \infty} P(C_N \leq zu^2) \\ &\stackrel{(2.2)}{\leq} \overline{\lim}_{N \rightarrow \infty} P(g(0)(1 - \delta)u \leq L_{[zu^2]}). \end{aligned} \quad (2.3)$$

But by (1.25) we have $P(g(0)(1 \pm \delta)u \leq L_{[zu^2]}) = P(\gamma_{g(0)u(1 \pm \delta)} \leq zu^2)$ and thus from (2.3), (1.24) (using that $u \rightarrow \infty$ as $N \rightarrow \infty$) and the continuity of the limit law (see (1.22)) we get:

$$\begin{aligned} \mathbb{P}(\zeta(\frac{1}{\sqrt{d+1}}) \leq \frac{z}{(g(0)(1 + \delta))^2}) &\leq \overline{\lim}_{N \rightarrow \infty} P(C_N \leq zu^2) \\ &\leq \mathbb{P}(\zeta(\frac{1}{\sqrt{d+1}}) \leq \frac{z}{(g(0)(1 - \delta))^2}). \end{aligned}$$

Once again by the continuity of the law of $\zeta(\cdot)$ we can now let $\delta \downarrow 0$ to get $\lim_{N \rightarrow \infty} P(C_N \leq zu^2) = \mathbb{P}(\zeta(\frac{1}{\sqrt{d+1}}) \leq \frac{z}{g(0)^2})$ and (2.1) then follows by the scaling relation (1.21). \square

Next we prove the weak convergence of the point process of vertices covered last (recall the definition of this process from (0.2)).

Corollary 2.2 (Convergence of point process of vertices covered last). ($z \in \mathbb{R}$) *In the topology of point processes*

$$\begin{aligned} \mathcal{N}_N^z &\text{ converges weakly to a Poisson point process} \\ &\text{on } (\mathbb{R}/\mathbb{Z})^d \times \mathbb{R} \text{ of intensity } \exp(-z)\lambda, \end{aligned} \quad (2.4)$$

where λ denotes Lebesgue measure on $(\mathbb{R}/\mathbb{Z})^d \times \{0\}$.

Proof. Let $\mathcal{M}_N^z = \sum_{x \in \mathbb{T}_N \times \{0\}} 1_{\{L_{H_x} \geq N^d u(z)\}}$. We will first show (2.4) with \mathcal{M}_N^z in place of \mathcal{N}_N^z . By Kallenberg's theorem (see Proposition 3.22, p. 156 of [14]) to show (2.4) with \mathcal{M}_N^z in place of \mathcal{N}_N^z it suffices to check that for all I in

$$\mathcal{J} = \{I : I \text{ a union of products of open intervals in } (\mathbb{R}/\mathbb{Z})^d \times \mathbb{R}, \lambda(I) > 0\},$$

the following two statements hold:

$$\lim_{N \rightarrow \infty} E\mathcal{M}_N^z(I) = \exp(-z)\lambda(I), \quad (2.5)$$

$$\lim_{N \rightarrow \infty} P(\mathcal{M}_N^z(I) = 0) = \exp(-e^{-z}\lambda(I)). \quad (2.6)$$

For any $I \in \mathcal{J}$ define $F'_N = NI \cap \mathbb{T}_N \times \{0\}$. Note that because of the special form of I (recall $\lambda(I) > 0$) we have

$$|F'_N| \rightarrow \infty \text{ and } \frac{|F'_N|}{|\mathbb{T}_N \times \{0\}|} \rightarrow \lambda(I) \text{ as } N \rightarrow \infty. \quad (2.7)$$

To show (2.6) we note that

$$P(\mathcal{M}_N^z(I) = 0) = P(L_{C'_N} < N^d u(z)), \quad (2.8)$$

where $C'_N = C_{F'_N}$ and $u(z) = u_{\mathbb{T}_N \times \{0\}}(z)$ (recall the notation from (1.43)). Let $u'_N = u_{F'_N}$ and note that $u(z) = u'_N(z - \log \frac{|F'_N|}{|\mathbb{T}_N \times \{0\}|})$ so that for all $a > 0$ and $N \geq c(I, z, a)$ we have

$$u'_N(z - a - \log \lambda(I)) \stackrel{(2.7)}{<} u(z) \stackrel{(2.7)}{<} u'_N(z + a - \log \lambda(I)). \quad (2.9)$$

Therefore using (0.5) with F'_N and C'_N in the place of F_N and C_{F_N} we get

$$\begin{aligned} \exp(-e^{-z+a} \lambda(I)) &\stackrel{(0.5)}{=} \underline{\lim}_{N \rightarrow \infty} P(L_{C'_N} \leq N^d u'_N(z - a - \log \lambda(I))) \\ &\stackrel{(2.8), (2.9)}{\leq} \overline{\lim}_{N \rightarrow \infty} P(\mathcal{M}_N^z(I) = 0) \\ &\stackrel{(2.8), (2.9)}{\leq} \overline{\lim}_{N \rightarrow \infty} P(L_{C'_N} \leq N^d u'_N(z + a - \log \lambda(I))) \\ &\stackrel{(0.5)}{=} \exp(-e^{-z-a} \lambda(I)), \end{aligned}$$

so that so letting $a \downarrow 0$ we find (2.6).

It remains to check (2.5). Note that

$$E\mathcal{M}_N^z(I) \stackrel{(1.25)}{=} \sum_{x \in F'_N} P(\gamma_{N^d u(z)} \leq H_x). \quad (2.10)$$

Let us now record (for use now and later) that

$$\begin{aligned} &\text{for } F \subset \mathbb{T}_N \times [-\frac{N}{2}, \frac{N}{2}] \text{ with } |F| \geq 2, z \in [-\frac{1}{2} \log |F|, -\frac{1}{2} \log |F|], \\ &\quad a \in (0, \frac{1}{10}], \delta = (\log N)^{-3/2} \text{ and } N \geq c(a) \text{ we have} \\ &\frac{2}{\sqrt{N}} \leq u_F(z - a) \leq u_F(z)(1 - \delta) \leq u_F(z)(1 + \delta) \leq u_F(z + a) \leq c \log N. \end{aligned} \quad (2.11)$$

Thus for any $a \in (0, \frac{1}{10}]$ and $N \geq c(z, a)$ it follows from (1.23) (note that $\frac{1}{2} > \delta \geq c_0 \frac{r_N}{h_N}$ for $N \geq c$, with δ as in (2.11)) that $P(D_{[K_N u(z+a)]} < \gamma_{N^d u(z)}), P(D_{[K_N u(z-a)]} \geq \gamma_{N^d u(z)}) \leq ce^{-cN^c}$ and thus

$$\begin{aligned} P(D_{[K_N u(z+a)]} < H_x) - ce^{-cN^c} &\leq P(\gamma_{N^d u(z)} \leq H_x) \\ &\leq P(D_{[K_N u(z-a)]} < H_x) + ce^{-cN^c}. \end{aligned} \quad (2.12)$$

Now since $\{D_k < H_x\} \stackrel{(1.1)}{=} \{x \notin X(0, D_k)\}$ for any $k \geq 1$ we have

$$P(D_{[K_N u(z \pm a)]} < H_x) = P(x \notin X(0, D_{[K_N u(z \pm a)]})). \quad (2.13)$$

We can now use (0.6) twice to get that if $a \in (0, \frac{1}{10}]$ and $N \geq c(z, a)$ then

$$\begin{aligned} P(x \notin X(0, D_{[K_N u(z-a)]})) &\leq Q_0(x \notin \mathcal{I}^{u(z-2a)}) + cN^{-3d}, \\ P(x \notin X(0, D_{[K_N u(z+a)]})) &\geq Q_0(x \notin \mathcal{I}^{u(z+2a)}) - cN^{-3d}, \end{aligned} \quad (2.14)$$

where we have also used (2.11) with $u(z \pm a)$ in the place of z , that $\delta \geq c_2 \frac{r_N}{h_N}$ (δ as in (2.11)) for $N \geq c$, the fact that $\mathcal{I}^{(1 \pm \delta)u_F(z \pm a)}$ has the same law under the coupling Q_1 as under the canonical probability Q_0 and (1.34). But $Q_0(x \notin \mathcal{I}^{u(z \pm 2a)}) = \frac{\exp(-z \mp 2a)}{|\mathbb{T}_N \times \{0\}|}$ by (1.37), so combining (2.12), (2.13) and (2.14) we get that if $N \geq c(z, a)$ then

$$\frac{\exp(-z - 2a)}{|\mathbb{T}_N \times \{0\}|} - cN^{-3d} \leq P(\gamma_{N^{d_{u(z)}}} \leq H_x) \leq \frac{\exp(-z + 2a)}{|\mathbb{T}_N \times \{0\}|} + cN^{-3d}.$$

So by (2.7) and (2.10) and we have that for all $z \in \mathbb{R}$ and $a \in (0, \frac{1}{10}]$

$$\lambda(I) \exp(-z - 2a) \leq \overline{\lim}_{N \rightarrow \infty} E\mathcal{M}_N^z(I) \leq \lambda(I) \exp(-z + 2a).$$

We simply have to take $a \downarrow 0$ to get (2.5). This completes the proof of (2.4) with \mathcal{M}_N^z in place of \mathcal{N}_N^z . Thus the limit of the Laplace functionals of \mathcal{M}_N^z is the map $f \rightarrow \exp(-e^{-z} \int (1 - e^{-f(x)}) \lambda(dx))$, which is continuous in z . By using the inequality $\mathcal{M}_N^{z+\delta} \stackrel{(0.2)}{\leq} \mathcal{N}_N^z \stackrel{(0.2)}{\leq} \mathcal{M}_N^z$, valid for any $z \in \mathbb{R}, \delta > 0$ and $N \geq 1$, and letting $\delta \downarrow 0$ we see that the limit of the Laplace functionals of \mathcal{N}_N^z is $f \rightarrow \exp(-e^{-z} \int (1 - e^{-f(x)}) \lambda(dx))$ as well, and therefore (2.4) holds. \square

Finally we derive Corollary 2.3 from Corollary 2.2.

Corollary 2.3 (Last vertices to be hit are far apart). *Let the random vector $(Y_1, Y_2, \dots, Y_{|\mathbb{T}_N \times \{0\}|})$ be the vertices of $\mathbb{T}_N \times \{0\}$ ordered by their entrance time with Y_1 being hit last, so that:*

$$C_{\mathbb{T}_N \times \{0\}} = H_{Y_1} > H_{Y_2} > \dots > H_{Y_{|\mathbb{T}_N \times \{0\}|}} = H_{\mathbb{T}_N \times \{0\}}.$$

Then for all $k \geq 2$

$$\lim_{\delta \rightarrow 0} \overline{\lim}_{N \rightarrow \infty} P(\exists 1 \leq i < j \leq k \text{ such that } d(Y_i, Y_j) \leq \delta N) = 0, \quad (2.15)$$

or in other words for large N the last k vertices of $\mathbb{T}_N \times \{0\}$ to be hit are separated, at typical distance of order N .

Proof. For $z \in \mathbb{R}$, Corollary 2.2 says that \mathcal{N}_N^z converges weakly to \mathcal{N}^z , a Poisson point process on $(\mathbb{R}/\mathbb{Z})^d \times \mathbb{R}$ of intensity $\exp(-z)\lambda$ (λ as in Corollary 2.2). Note that for any $z \in \mathbb{R}$ and $\delta > 0$ the limsup of the probability in (2.15) is bounded above by

$$\overline{\lim}_{N \rightarrow \infty} P(\exists x \neq y \in \text{Supp}(\mathcal{N}_N^z), d(x, y) \leq \delta) + \mathbb{P}(\mathcal{N}^z((\mathbb{R}/\mathbb{Z})^d \times \mathbb{R}) < k), \quad (2.16)$$

where we have used that $P(\mathcal{N}_N^z((\mathbb{R}/\mathbb{Z})^d \times \mathbb{R}) < k) \rightarrow \mathbb{P}(\mathcal{N}^z((\mathbb{R}/\mathbb{Z})^d \times \mathbb{R}) < k)$ as $N \rightarrow \infty$. Let $f : (\mathbb{R}/\mathbb{Z})^d \times \mathbb{R} \rightarrow [0, 1]$ be a continuous function such that $f(x) = 1$ if $d(0, x) \leq \delta$ and $f(x) = 0$ if $d(0, x) \geq 2\delta$. Then

$$\begin{aligned} \lim_{\delta \rightarrow 0} \overline{\lim}_{N \rightarrow \infty} P(\exists x \neq y \in \text{Supp}(\mathcal{N}_N^z) \text{ s.t. } d(x, y) \leq \delta) &\leq \\ \lim_{\delta \rightarrow 0} \overline{\lim}_{N \rightarrow \infty} P(\sum_{x, y \in \text{Supp}(\mathcal{N}_N^z), x \neq y} f(x - y) \geq 1) &\leq \\ \lim_{\delta \rightarrow 0} \overline{\lim}_{N \rightarrow \infty} E(\sum_{x, y \in \text{Supp}(\mathcal{N}_N^z), x \neq y} f(x - y)), & \end{aligned} \quad (2.17)$$

where we have used Markov's inequality going from the middle to the last line. Consider the sum $\sum_{x,y \in \text{Supp}(\mathcal{N}_N^z), x \neq y} f(x-y) = \mathcal{N}_N^z \otimes \mathcal{N}_N^z(f(\cdot - \cdot)) - f(0)\mathcal{N}_N^z((\mathbb{R}/\mathbb{Z})^d \times \mathbb{R})$. Note that $\mathcal{N}_N^z \otimes \mathcal{N}_N^z$ tends weakly to $\mathcal{N}^z \otimes \mathcal{N}^z$ so that

$$\lim_{N \rightarrow \infty} E \left[\sum_{x,y \in \text{Supp}(\mathcal{N}_N^z), x \neq y} f(x-y) \right] = \mathbb{E} \left[\sum_{x,y \in \text{Supp}(\mathcal{N}^z), x \neq y} f(x-y) \right]. \quad (2.18)$$

By Proposition 13.1.VII p. 280 of [7] the local Palm distribution for \mathcal{N}^z at $x \in (\mathbb{R}/\mathbb{Z})^d \times \mathbb{R}$ is the distribution of $\mathcal{N}^z + \delta_x$. Therefore (by e.g. Proposition 13.1.IV p. 273 of [7]) the right hand side of (2.18) equals

$$\begin{aligned} e^{-z} \int \mathbb{E} [(\mathcal{N}^z + \delta_x)(1_{\{y \neq x\}} f(x-y))] \lambda(dx) &= e^{-2z} \iint f(x-y) \lambda(dy) \lambda(dx) \\ &\leq c(z) \delta^d. \end{aligned} \quad (2.19)$$

Combining (2.18) and (2.19) we get that the right-hand side of (2.17) equals zero, and therefore from (2.16) we see that for all $z \in \mathbb{R}$

$$\lim_{\delta \rightarrow 0} \overline{\lim}_{N \rightarrow \infty} P(\exists 1 \leq i < j \leq k \text{ such that } d(Y_i, Y_j) \leq \delta N) \leq \mathbb{P}(\mathcal{N}^z((\mathbb{R}/\mathbb{Z})^d \times \mathbb{R}) < k). \quad (2.20)$$

But if we take $z \rightarrow -\infty$ then the right hand side of (2.20) tends to zero, so (2.15) follows. \square

3 Convergence to Gumbel

We now turn to the proof of Theorem 0.1. Intuitively speaking (0.5) says that “ $L_{C_{F_N}}$ is approximately distributed as a Gumbel random variable with location $N^d g(0) \log |F_N|$ and scale $N^d g(0)$ ” (recall that a Gumbel random variable with location μ and scale β has cumulative distribution function $\exp(-e^{-(x-\mu)/\beta})$ and that the standard Gumbel distribution has location 0 and scale 1). The first step of the proof is to use (1.23) to reduce this to the statement “the number of excursions needed to cover F_N is approximately Gumbel distributed with location $K_N g(0) \log |F_N|$ and scale $K_N g(0)$ ”. To prove this latter statement we want to use the coupling result Theorem 4.1 from Section 4 that couples the trace of the random walk X with random interlacements, and apply (1.45) (i.e. Theorem 0.1 of [4]) which gives the corresponding distributional limit result in the random interlacements model. It is however not immediately obvious how this might be done because Theorem 4.1 only couples the trace of X in several *separated* “local boxes” of side length $N^{1-\varepsilon}$, $0 < \varepsilon < 1$ centred in $\mathbb{T}_N \times [-\frac{N}{2}, \frac{N}{2}]$ with random interlacements, and in general it will not be possible to cover F_N by a collection of such local boxes. We are able to deal with this problem by splitting the sequence F_N into two subsequences, one with F_N that are small in the sense that $|F_N| \leq N^{1/8}$ and one with F_N that are big in the sense that $|F_N| > N^{1/8}$. In the first case (small F_N) we are able to apply Theorem 4.1 and (1.45) to get that the number of excursions needed to cover F_N is approximately Gumbel distributed with appropriate parameters. Moreover we are able to reduce the second case (big F_N) to the first case.

We now state Proposition 3.1 which deals the first case. Recall (1.4), (1.43) and (1.44) for notation.

Proposition 3.1 (Gumbel for “small” F). ($d \geq 2$) For any $\theta \in (0, \frac{1}{10}]$, $N \geq 1$ and $F \subset \mathbb{T}_N \times [-\frac{N}{2}, \frac{N}{2}]$ such that $N^{1/8} \geq |F| \geq c(\theta)$ we have:

$$\sup_{z \in [-\frac{1}{2} \log |F|, \frac{1}{2} \log |F|], l \in [-N, N]} |P_{q_l}(F \subset X(0, D_{[K_N u_F(z)]})) - G(z)| \leq \theta. \quad (3.1)$$

We postpone the proof of Proposition 3.1 until after the proof of Theorem 0.1 and instead state Proposition 3.2 which deals with the second case.

Proposition 3.2 (Gumbel for “big” F_N). ($d \geq 2$) Let F_N be a sequence of sets as in Theorem 0.1 with the additional requirement that $|F_N| > N^{1/8}$ for all $N \geq 1$. Then for all $z \in \mathbb{R}$ (recall that u_N is a shorthand for u_{F_N}):

$$P(F_N \subset X(0, D_{[K_N u_N(z)]})) \rightarrow G(z) \text{ as } N \rightarrow \infty. \quad (3.2)$$

The proof of Proposition 3.2 is also postponed until after the proof of Theorem 0.1, which we now start.

Proof of Theorem 0.1. We write C_N for C_{F_N} . By (1.25) we have $P(L_{C_N} < N^d u_N(z)) = P(C_N < \gamma_{N^d u_N(z)})$ for all $z \geq -\frac{1}{2} \log |F_N|$. Also note that for all $z \in \mathbb{R}$, $a \in (0, \frac{1}{10}]$ and $N \geq c(z, a, (F_N)_{N \geq 1})$ it holds that

$$\begin{aligned} P(C_N < \gamma_{N^d u_N(z)}) &\geq P(C_N \leq D_{[u_N(z-a)K_N]}) - P(D_{[u_N(z-a)K_N]} \geq \gamma_{N^d u_N(z)}), \\ P(C_N < \gamma_{N^d u_N(z)}) &\leq P(C_N \leq D_{[u_N(z+a)K_N]}) + P(D_{[u_N(z+a)K_N]} < \gamma_{N^d u_N(z)}). \end{aligned}$$

Now using that $P(C_N \leq D_{[u_N(z \pm a)K_N]}) \stackrel{(1.1)}{=} P(F_N \subset X(0, D_{[u_N(z \pm a)K_N]}))$ and (2.11) with $F = F_N$ (similarly to under (2.11) but with $z \pm a$ in place of z) it follows from the above and two applications of (1.23) that for all $z \in \mathbb{R}$ and $a \in (0, \frac{1}{10}]$

$$\begin{aligned} \lim_{N \rightarrow \infty} P(F_N \subset X(0, D_{[u_N(z-a)K_N]})) &\leq \overline{\lim}_{N \rightarrow \infty} P(L_{C_N} < N^d u_N(z)) \\ &\leq \lim_{N \rightarrow \infty} P(F_N \subset X(0, D_{[u_N(z+a)K_N]})). \end{aligned} \quad (3.3)$$

But by splitting the sequence F_N into two subsequences and applying Proposition 3.1 (recall that $P = P_{q_0}$) and Proposition 3.2 it follows that $\lim_{N \rightarrow \infty} P(F_N \subset X(0, D_{[u_N(z \pm a)K_N]})) = G(z \pm a)$. We can thus replace the right- and left-hand sides of (3.3) with $G(z + a)$ and $G(z - a)$ respectively, and then let $a \downarrow 0$ and use the continuity of G to conclude that $\lim_{N \rightarrow \infty} P(L_{C_N} < N^d u_N(z)) = G(z)$ for all $z \in \mathbb{R}$, and therefore that (0.5) holds. \square

We now turn to the proof of Proposition 3.1 which deals with “small” sets F_N . It turns out that such small sets can be chopped into pieces S_1, S_2, \dots, S_k such that each individual piece is contained in a local box of side length $N^{1/2}$, and is separated from the other pieces by a distance of at least $|F|^3$. The separation will allow us to apply Theorem 4.1 to conclude that the traces left by X on each S_i are approximated by k independent random interlacements, and thus that the number of excursions needed to cover F is approximately K_N times $\max_k \tilde{C}_{S_k}$, the maximum of the cover levels of the S_k by the k random interlacements. Using (1.38) we will be able to assemble the pieces (placing them suitably far apart) into a single random interlacement, so that $\max_k \tilde{C}_{S_k}$ is close in distribution to the cover level $\tilde{C}_{\tilde{F}}$ of a set \tilde{F} which consists of the reassembled

pieces S_1, S_2, \dots, S_k and thus has the same cardinality as F . It will then be straightforward to prove that $\tilde{C}_{\tilde{F}}$ (and thus $\max_k \tilde{C}_{S_k}$) is approximately distributed as a Gumbel random variable with location $g(0) \log |F|$ and scale $g(0)$ by applying (1.45). This in turn will imply that the number of excursions needed to cover F is approximately distributed as a Gumbel random variable with location $K_N g(0) \log |F|$ and scale $K_N g(0)$, which is essentially speaking what Proposition 3.1 claims.

Proof of Proposition 3.1. Construct a graph (F, E_F) with vertices in F and edge set E_F such that $\{a, b\} \in E_F$ iff $a, b \in F$ and $d_\infty(a, b) \leq |F|^3$. Let S_1, S_2, \dots, S_k be the connected components of (F, E_F) and let $x_1, \dots, x_k \in E_N$ be arbitrarily selected $x_i \in S_i, i = 1, \dots, k$. Then for each i and for all $y \in S_i$ we have $d_\infty(y, x_i) \leq |F| \times |F|^3 = |F|^4 \leq N^{1/2}$, where the last inequality follows by our assumption on the cardinality of F . Therefore $S_i \subset B(x_i, N^{1/2})$ for all i , and if $i \neq j$ we have $d_\infty(S_i, S_j) > |F|^3$.

We now apply Theorem 4.1 with $\varepsilon = \frac{1}{2}$, $u = u_F(z)$, δ as in (2.11) (using also (2.11) with $a = \theta/4$, similarly to under (2.14)) and l in the place of z to get that for any $l \in [-N, N]$, $z \in [-\frac{1}{2} \log |F|, \frac{1}{2} \log |F|]$ and $|F| \geq c(\theta)$

$$\begin{aligned} \prod_{i=1}^k Q_0(S_i - x_i \subset \mathcal{I}^{u_F(z - \frac{\theta}{4})}) - \frac{\theta}{4} &\leq P_{q_l}(F \subset X(0, D_{[K_N u_F(z)]})) \\ &\leq \prod_{i=1}^k Q_0(S_i - x_i \subset \mathcal{I}^{u_F(z + \frac{\theta}{4})}) + \frac{\theta}{4}, \end{aligned} \quad (3.4)$$

where we have also used that $N \geq |F|^8$ and (when $k > 1$) that $\sum_{i \neq j} \mathcal{E}(S_i, S_j) \leq \sum_{i \neq j} c \frac{|S_i||S_j|}{|F|^3} \leq \frac{c}{|F|}$ by (1.13) and $d(S_i, S_j) \geq c|F|^3$. Consider the sets $\tilde{S}_i = (S_i - x_i) + i \cdot \exp(N)e_1 \subset \mathbb{Z}^{d+1}$, where $S_i - x_i$ is viewed as a subset of \mathbb{Z}^{d+1} and $e_1 = (1, 0, \dots, 0) \in \mathbb{Z}^{d+1}$. Let $\tilde{F} = \bigcup_{i=1}^k \tilde{S}_i$. By (1.38) and (1.5) the following holds for $z \in [-\frac{1}{2} \log |F|, \frac{1}{2} \log |F|]$:

$$|Q_0(\tilde{F} \subset \mathcal{I}^{u_F(z \pm \frac{\theta}{4})}) - Q_0(\tilde{S}_1 \subset \mathcal{I}^{u_F(z \pm \frac{\theta}{4})})Q_0(\tilde{F} \setminus \tilde{S}_1 \subset \mathcal{I}^{u_F(z \pm \frac{\theta}{4})})| \leq c \log |F| \frac{|\tilde{S}_1||\tilde{F}|}{\exp(N)}.$$

Applying (1.38) another $k - 1$ times and using the triangle inequality we get that

$$\left| Q_0(\tilde{F} \subset \mathcal{I}^{u_F(z \pm \frac{\theta}{4})}) - \prod_{i=1}^k Q_0(S_i - x_i \subset \mathcal{I}^{u_F(z \pm \frac{\theta}{4})}) \right| \leq c \log |F| \frac{|F|^2}{\exp(N)} \leq \frac{\theta}{4} \quad (3.5)$$

holds for all $|F| \geq c(\theta)$ and $z \in [-\frac{1}{2} \log |F|, \frac{1}{2} \log |F|]$ (using also that $Q_0(S_i - x_i \subset \mathcal{I}^{u_F(z \pm \frac{\theta}{4})}) \stackrel{(1.34)}{=} Q_0(\tilde{S}_i \subset \mathcal{I}^{u_F(z \pm \frac{\theta}{4})})$). Now finally we apply (1.45), using that $|F| = |\tilde{F}|$, to get that for all $|F| \geq c(\theta)$ and $z \geq -\frac{1}{2} \log |F|$:

$$\left| Q_0(\tilde{F} \subset \mathcal{I}^{u_F(z \pm \frac{\theta}{4})}) - G(z \pm \frac{\theta}{4}) \right| \leq \frac{\theta}{4}. \quad (3.6)$$

Combining (3.4), (3.5) and (3.6) with the fact that $|G(z) - G(z \pm \frac{\theta}{4})| \leq \frac{\theta}{4}$ for all z , we get (3.1). \square

Next we prove Proposition 3.2, which deals with “big” sets F_N . In this case we will consider for some $0 < \rho < 1$ the set $F_N^\rho = F_N \setminus X(0, D_{[(1-\rho)K_N u_N(0)]})$, i.e. the subset of F_N left uncovered by a fraction $1 - \rho$ of the typical number of excursions $K_N u_N(0)$ needed to cover F_N , and show that the cardinality of F_N^ρ concentrates around its typical value

which turns out to be $|F_N|^\rho$ (this is the content of Lemma 3.3). To do this we will once again split F_N into pieces S_1, \dots, S_n that are contained in local boxes (but this time they will not in general be far apart). Using the coupling Theorem 4.1 for one i at a time will allow us to use a random interlacement calculation to prove that $|F_N^\rho \cap S_i|$ concentrates around $\frac{|S_i|}{|F_N|} \times |F_N|^\rho$. A union bound will then ensure that $|F_N^\rho \cap S_i|$ concentrates around this value for all i at the same time with high probability, and thus that $|F_N^\rho|$ concentrates around $|F_N|^\rho$ with high probability.

Now for the ρ we pick (cf. (3.20)) we will have $|F_N|^\rho \leq ((2N+1)N^d)^\rho \leq N^{1/8}$ so that (with high probability) F_N^ρ is a “small set” in the sense of Proposition 3.1. It will turn out that the excursions after the $[(1-\rho)K_N u_N(0)]$ -th departure are “almost” independent of F_N^ρ and therefore we will be able to apply Proposition 3.1 to the set F_N^ρ to prove that the number of additional excursions needed to cover it is approximately a Gumbel random variable with location $K_N g(0) \log |F_N^\rho| \approx K_N g(0) \log |F_N|^\rho = K_N g(0) \rho \log |F_N|$ and scale $K_N g(0)$. Adding to this random variable $(1-\rho)K_N u_N(0) = (1-\rho)K_N \log |F_N|$, the approximate number of excursions that reduced F_N to F_N^ρ , we find that the number of excursions needed to cover F_N is approximately distributed as a Gumbel random variable with location $K_N g(0) \log |F_N|$ and scale $K_N g(0)$, which is essentially speaking what Proposition 3.2 claims.

Proof of Proposition 3.2. Fix a $z \in \mathbb{R}$. Define for any $\rho \in (0, \frac{1}{4}]$ and $N \geq c$

$$r(\rho) = r(\rho, N) = [(1-\rho)K_N u_N(0)] \text{ and } F_N^\rho = F_N \setminus X(0, D_{r(\rho)}). \quad (3.7)$$

Because $\{F_N \subset X(0, D_{[K_N u_N(z)]})\} = \{F_N^\rho \subset X(R_{r(\rho)+1}, D_{[K_N u_N(z)]})\}$ for $N \geq c$ it suffices to show that for some $\rho \in (0, \frac{1}{4}]$

$$P(F_N^\rho \subset X(R_{r(\rho)+1}, D_{[K_N u_N(z)]})) \rightarrow G(z) \text{ as } N \rightarrow \infty. \quad (3.8)$$

For each $\lambda \in (0, \frac{1}{100})$ and $N \geq 1$ we define the collection of “good sets” by:

$$\mathcal{G}_{N,\lambda} = \{F' \subset F_N : (1-\lambda)|F_N|^\rho \leq |F'| \leq (1+\lambda)|F_N|^\rho\}. \quad (3.9)$$

To show (3.8) we will use the following lemma:

Lemma 3.3. *For all $\lambda \in (0, \frac{1}{100})$ we have $\lim_{N \rightarrow \infty} P(F_N^\rho \in \mathcal{G}_{N,\lambda}) = 1$.*

Proof. Let

$$\varepsilon = \frac{1}{3(d+1)} \frac{1}{16} \rho. \quad (3.10)$$

For each N we can select $x_1, \dots, x_{n(N)} \in \mathbb{T}_N \times [-\frac{N}{2}, \frac{N}{2}]$, $n(N) \leq cN^{(d+1)\varepsilon}$ and partition F_N into disjoint non-empty sets S_N^1, \dots, S_N^n such that $S_N^i \subset B(x_i, N^{1-\varepsilon})$ for each i and $F_N = \bigcup_{i=1}^n S_N^i$. Consider for each $i = 1, \dots, n(N)$ the events

$$\begin{aligned} E_i^+ &= \left\{ \left| S_N^i \setminus X(0, D_{r(\rho)}) \right| \geq (1 + \frac{\lambda}{2}) a_i |F_N|^\rho + \frac{\lambda}{2t} \sqrt{a_i} |F_N|^\rho \right\} \text{ and} \\ E_i^- &= \left\{ \left| S_N^i \setminus X(0, D_{r(\rho)}) \right| \leq (1 - \frac{\lambda}{2}) a_i |F_N|^\rho - \frac{\lambda}{2t} \sqrt{a_i} |F_N|^\rho \right\}, \end{aligned} \quad (3.11)$$

where $a_i = \frac{|S_N^i|}{|F_N|}$ and $t = \sum_{i=1}^{n(N)} \sqrt{a_i}$. Because the right-hand sides of the inequalities in the events in (3.11) sum up to $(1 \pm \lambda)|F_N|^\rho$ we have:

$$\begin{aligned} P(|F_N^\rho| \geq (1+\lambda)|F_N|^\rho) &\leq \sum P(E_i^+) \leq cN^{(d+1)\varepsilon} \sup_i P(E_i^+), \\ P(|F_N^\rho| \leq (1-\lambda)|F_N|^\rho) &\leq \sum P(E_i^-) \leq cN^{(d+1)\varepsilon} \sup_i P(E_i^-). \end{aligned} \quad (3.12)$$

We therefore wish to bound $P(E_i^\pm)$. To this end let

$$u_\pm(\lambda) = g(0)\{(1-\rho)\log|F_N| + \log \frac{1}{1 \mp \lambda/2}\} \text{ and } u = (1-\rho)u_N(0), \quad (3.13)$$

and note that we can apply Theorem 4.1 with $k = 1$, ε as in (3.10), u as in (3.13), δ as in (2.11) (using that $u_-(\lambda) \leq (1-\rho)u_N(-\frac{\lambda}{4}) \leq (1-\rho)u_N(\frac{\lambda}{4}) \leq u_+(\lambda)$, $u = (1-\rho)u_N(0)$, so that $[uK_N] = r(\rho)$, and (2.11) with $a = \lambda/4$ similarly to under (2.14) and above (3.4)) once for each i to show that if $N \geq c(\lambda)$ then

$$\begin{aligned} P(E_i^+) &\leq Q_0(|S_N^i \setminus \mathcal{I}^{u_-}| \geq (1 + \frac{\lambda}{2})a_i|F_N|^\rho + \frac{\lambda}{2t}\sqrt{a_i}|F_N|^\rho) + cN^{-3d}, \\ P(E_i^-) &\leq Q_0(|S_N^i \setminus \mathcal{I}^{u_+}| \leq (1 - \frac{\lambda}{2})a_i|F_N|^\rho - \frac{\lambda}{2t}\sqrt{a_i}|F_N|^\rho) + cN^{-3d}, \end{aligned} \quad (3.14)$$

where we view S_N^i as a subset of \mathbb{Z}^{d+1} and have also used (1.34). Note that

$$\mathbb{E}^{Q_0}(|S_N^i \setminus \mathcal{I}^{u_\mp}|) = \sum_{x \in S_N^i} Q_0(x \notin \mathcal{I}^{u_\mp}) \stackrel{(1.37), (3.13)}{=} (1 \pm \frac{\lambda}{2})a_i|F_N|^\rho. \quad (3.15)$$

Thus the probabilities on the right hand sides of (3.14) are bounded above by:

$$Q_0(|S_N^i \setminus \mathcal{I}^{u_\mp}| - \mathbb{E}^{Q_0}(|S_N^i \setminus \mathcal{I}^{u_\mp}|)| \geq \frac{\lambda}{2t}\sqrt{a_i}|F_N|^\rho). \quad (3.16)$$

Using the Chebyshev inequality we see that (3.16) is bounded above by:

$$\frac{\mathbb{E}^{Q_0}(|S_N^i \setminus \mathcal{I}^{u_\mp}|^2) - (\mathbb{E}^{Q_0}(|S_N^i \setminus \mathcal{I}^{u_\mp}|))^2}{\frac{\lambda^2}{4t^2}a_i|F_N|^{2\rho}} \quad (3.17)$$

We thus wish to bound $\mathbb{E}^{Q_0}(|S_N^i \setminus \mathcal{I}^{u_\mp}|^2)$. Note that

$$\begin{aligned} \mathbb{E}^{Q_0}(|S_N^i \setminus \mathcal{I}^{u_\mp}|^2) &= \sum_{x, y \in S_N^i} Q_0(x, y \notin \mathcal{I}^{u_\mp}) \stackrel{(3.15)}{\leq} ca_i|F_N|^\rho \\ &\quad + \sum_{x \in S_N^i} \sum_{y \in S_N^i, y \neq x} Q_0(x, y \notin \mathcal{I}^{u_\mp}). \end{aligned} \quad (3.18)$$

Now using the translation invariance of \mathcal{I}^{u_\mp} (cf. (1.34)) and Lemma 1.5 (with $a = 10000N$ say) we get that for $N \geq c$:

$$\begin{aligned} &\sum_{x \in S_N^i, y \neq x} Q_0(x, y \notin \mathcal{I}^{u_\mp}) \stackrel{(3.13)}{\leq} \\ &(1 \pm \frac{\lambda}{2})^2|F_N|^{2\rho-2} \{ |S_N^i|^2 + c \log|F_N| |S_N^i|^{\frac{2}{d+1}+1} \} + c|S_N^i||F_N|^{\rho-1} \stackrel{(3.15)}{\leq} \\ &\left(\mathbb{E}^{Q_0}(|S_N^i \setminus \mathcal{I}^{u_\mp}|) \right)^2 + ca_i \log|F_N| |F_N|^{2\rho-\frac{d-1}{d+1}} + ca_i|F_N|^\rho \stackrel{\rho \leq \frac{1}{4}}{\leq} \\ &\left(\mathbb{E}^{Q_0}(|S_N^i \setminus \mathcal{I}^{u_\mp}|) \right)^2 + ca_i|F_N|^\rho \end{aligned} \quad (3.19)$$

Combining (3.19) with (3.16), (3.17) and (3.18) we find from (3.14) that $P(E_i^\pm) \leq c(\lambda)t^2|F_N|^{-\rho} + cN^{-3d} \stackrel{t \geq 1}{\leq} c(\lambda)t^2N^{-\frac{1}{8}\rho}$ and thus from (3.12) that

$$P(\{(1-\lambda)|F_N|^\rho \leq |F_N^\rho| \leq (1+\lambda)|F_N|^\rho\}^c) \leq c(\lambda)N^{(d+1)\varepsilon}t^2N^{-\frac{1}{8}\rho} \leq c(\lambda)N^{-\frac{1}{16}\rho},$$

where to get the last inequality we have used that $t = \sum_{i=1}^{n(N)} \sqrt{a_i} \leq n(N) \leq cN^{(d+1)\varepsilon}$ and (3.10). Thus we just have to let $N \rightarrow \infty$ and recall the definition (3.9) of $\mathcal{G}_{N,\lambda}$ to complete the proof of Lemma 3.3. \square

We now continue with the proof of (3.8). Fix

$$\rho = \frac{1}{16(d+1)} \in (0, \frac{1}{4}] \quad (3.20)$$

and write r in place of $r(\rho)$. Note that by the strong Markov property

$$\sum P(F_N^\rho \subset X(R_{r+1}, D_{[K_N u_N(z)]}), F_N^\rho \in \mathcal{G}_{N,\lambda}) = \sum P(F_N^\rho = F', X_{D_r} = x) P_x(F' \subset X(0, D_{[K_N u_N(z)]-r})). \quad (3.21)$$

where the sum is over all $x \in \partial_e \tilde{B}$ and $F' \in \mathcal{G}_{N,\lambda}$. We now need the following lemma:

Lemma 3.4. *For any $\lambda \in (0, \frac{1}{100})$ and $N \geq c(\lambda, z)$ we have that if $F' \in \mathcal{G}_{N,\lambda}$ and $x \in \partial_e \tilde{B}$ then*

$$|P_x(F' \subset X(0, D_{[K_N u_N(z)]-r})) - G(z)| \leq c\lambda. \quad (3.22)$$

Proof. By the strong Markov property we have

$$P_x(F' \subset X(0, D_{[K_N u_N(z)]-r})) = E_x[P_{X_{R_1}}(F' \subset X(0, D_{[K_N u_N(z)]-r}))]. \quad (3.23)$$

Let $x = (y, w)$, with $y \in \mathbb{T}_N$ and $w = \{-h_N, h_N\}$, and let $v \in \{-r_N, r_N\}$ with $wv > 0$. Then by (2.2) of [21] we have

$$\sup_{x' \in \mathbb{T}_N \times \{v\}} |P_x(X_{R_1} = x') - q_v(x')| \leq cN^{-5d}.$$

Thus for $N \geq c(\lambda)$ we have

$$|E_x[P_{X_{R_1}}(F' \subset X(0, D_{[K_N u_N(z)]-r}))] - P_{q_v}(F' \subset X(0, D_{[K_N u_N(z)]-r}))| \leq \lambda. \quad (3.24)$$

Note that $g(0)\{\rho \log |F_N| + z\} = g(0)\{\log |F'| + z + \log \frac{|F_N|^\rho}{|F'|}\}$ and since $F' \in \mathcal{G}_{N,\lambda}$ we have $-4\lambda \leq \log \frac{1}{1+\lambda} \leq \log \frac{|F_N|^\rho}{|F'|} \leq \log \frac{1}{1-\lambda} \leq 4\lambda$. Thus for all $N \geq c(z, \lambda)$ it holds that

$$K_N u_{F'}(z - 8\lambda) \stackrel{(3.7)}{\leq} [K_N u_N(z)] - r \stackrel{(3.7)}{\leq} K_N u_{F'}(z + 8\lambda),$$

and therefore also that

$$\begin{aligned} P_{q_v}(F' \subset X(0, D_{[K_N u_{F'}(z-8\lambda)]})) &\leq P_{q_v}(F' \subset X(0, D_{[K_N u_N(z)]-r})) \\ &\leq P_{q_v}(F' \subset X(0, D_{[K_N u_{F'}(z+8\lambda)]})). \end{aligned} \quad (3.25)$$

Now if $N \geq c(\lambda, z)$ then $|F'| \stackrel{(3.9), (3.20)}{\leq} cN^{\frac{1}{16}} \leq N^{\frac{1}{8}}$, $|F'| \stackrel{(3.9)}{\geq} cN^{\frac{1}{8}\rho} \geq c(\lambda)$ and $-\frac{1}{2} \log |F'| \leq z - 8\lambda \leq z + 8\lambda \leq \frac{1}{2} \log |F'|$. We can therefore use Proposition 3.1 with λ in the place of θ on the right- and left-hand sides of (3.25) to get that

$$G(z - 8\lambda) - \lambda \leq P_{q_v}(F' \subset X(0, D_{[K_N u_N(z)]-r})) \leq G(z + 8\lambda) + \lambda. \quad (3.26)$$

Now we simply have to combine (3.23), (3.24) and (3.26) with the inequality $|G(z \pm 8\lambda) - G(z)| \leq c\lambda$ to get (3.22). \square

We are now ready to complete the proof of Proposition 3.2. From (3.22) and (3.21) we see that if $\lambda \in (0, \frac{1}{100})$ and $N \geq c(\lambda, z)$ then

$$|P(F_N^\rho \subset X(R_{r+1}, D_{[K_N u_N(z)]}), F_N^\rho \in \mathcal{G}_{N,\lambda}) - P(F_N^\rho \in \mathcal{G}_{N,\lambda})G(z)| \leq c\lambda.$$

Letting $N \rightarrow \infty$ and using Lemma 3.3 we see that

$$|\lim_{N \rightarrow \infty} P(F_N^\rho \subset X(R_{r+1}, D_{[K_N u_N(z)]})) - G(z)| \leq c\lambda.$$

Now letting $\lambda \downarrow 0$ we get (3.8) and therefore the proof of Proposition 3.2 is complete. \square

We have now completely reduced the proofs Theorem 0.1 and its corollaries to the coupling Theorem 4.1.

4 Coupling random walk with random interlacements

In this section we state and prove the main coupling theorem, Theorem 4.1, which couples random walk X with random interlacements. More precisely, for some $\varepsilon \in (0, 1)$, $k \geq 1$ and suitably large N we select k vertices

$$x_1, \dots, x_k \in \mathbb{T}_N \times [-\frac{N}{2}, \frac{N}{2}] \text{ and } k \text{ non-empty sets} \\ S_1, S_2, \dots, S_k \subset B(0, N^{1-\varepsilon}) \text{ such that } S_i + x_i, i = 1, \dots, k \text{ are disjoint,} \quad (4.1)$$

and then construct, for appropriate u and δ , k independent pairs of random sets $\mathcal{I}_i^{u(1-\delta)} \cap S_i, \mathcal{I}_i^{u(1+\delta)} \cap S_i$, with the law of random interlacements at level $u(1-\delta)$, respectively $u(1+\delta)$, intersected with S_i , such that the following event holds with high probability (provided the $S_i + x_i$ have low mutual energy, see (1.9), for example if they are “far apart”):

$$\{\mathcal{I}_i^{u(1-\delta)} \cap S_i \subset (X(0, D_{[uK_N]} - x_i) \cap S_i \subset \mathcal{I}_i^{u(1+\delta)} \cap S_i \text{ for all } i\}. \quad (4.2)$$

A weaker version of the coupling which gave the upper inclusion for $k = 1$, *fixed* u and δ and large N is contained in [21] (a similar lower inclusion is contained implicitly in [20]). Theorem 4.1 improves on this by allowing u and δ to vary with N , by constructing $\mathcal{I}_i^{u(1\pm\delta)} \cap S_i$ such that they have the *joint* law of random interlacements at levels $u(1 \pm \delta)$ intersected with S_i (the naive way of combining the explicit coupling from [21] with the implicit coupling from [20] to get a double inclusion, as in (4.2), does not guarantee the “correct” joint law), and by coupling the trace in several sets $S_1 + x_1, \dots, S_k + x_k$. For more on why constructing $\mathcal{I}_i^{u(1\pm\delta)} \cap S_i$ such that they have the correct joint law is useful see Remark 6.11 (2).

The proof of Theorem 4.1 is divided into three steps: The first is to construct two independent Poisson processes of excursions (that is point processes on the space $\mathcal{T}_{\bar{B}}$ of intensity proportional to ν , see (1.32)) μ_1 and μ_2 , such that with high probability $(\mathcal{I}(\mu_1) - x_i) \cap S_i \subset (X(0, D_{[uK_N]} - x_i) \cap S_i \subset (\mathcal{I}(\mu_1) \cup \mathcal{I}(\mu_2) - x_i) \cap S_i$ for all i . This is done in Section 5; in the proof of Theorem 4.1 we invoke Corollary 5.3 for this step.

The second step is to exploit the fact that “the traces of a Poisson process of excursions on sets of low mutual energy are approximately independent” (if $k > 1$, otherwise this step is trivial) to construct k *independent* pairs of Poisson processes of excursions $\mu_1^i, \mu_2^i, i = 1, \dots, k$ such that with high probability $(\mathcal{I}(\mu_1^i) - x_i) \cap S_i = (\mathcal{I}(\mu_1) - x_i) \cap S_i$ and $(\mathcal{I}(\mu_1^i) \cup \mathcal{I}(\mu_2^i) - x_i) \cap S_i = (\mathcal{I}(\mu_1) \cup \mathcal{I}(\mu_2) - x_i) \cap S_i$ for $i = 1, \dots, k$. This is done in Lemma 4.2.

The third step is to construct from μ_1^i, μ_2^i the random sets $\mathcal{I}_i^{u(1-\delta)} \cap S_i, \mathcal{I}_i^{u(1+\delta)} \cap S_i$ from (4.2) such that with high probability $\mathcal{I}_i^{u(1-\delta)} \cap S_i \subset (\mathcal{I}(\mu_1^i) - x_i) \cap S_i$ and $(\mathcal{I}(\mu_1^i) \cup \mathcal{I}(\mu_2^i) - x_i) \cap S_i \subset \mathcal{I}_i^{u(1+\delta)} \cap S_i$. This is done mainly in Section 6; in the proof of Theorem 4.1 we invoke Proposition 6.1 for this step.

We thus postpone a large part of the work to Sections 5 and 6, and here only prove Theorem 4.1 conditionally on the results of these two sections. We now state the theorem.

Theorem 4.1. ($d \geq 2$) For any $k \geq 1$, $\varepsilon \in (0, 1)$, $N \geq c(\varepsilon)$, $z \in [-N, N]$, $1 \geq \delta \geq c_2 \frac{r_N}{h_N}$, u satisfying $uK_N \geq (\log N)^6$, and $x_1, \dots, x_k, S_1, \dots, S_k$ as in (4.1) we can construct on a space $(\Omega_1, \mathcal{A}_1, Q_1)$ an E_N -valued random walk X with law P_{q_z} , and an independent collection $((\mathcal{I}_i^{u(1-\delta)} \cap S_i, \mathcal{I}_i^{u(1+\delta)} \cap S_i))_{i=1}^k$ such that the i -th member of the collection has the (joint) law of $(\mathcal{I}^{u(1-\delta)} \cap S_i, \mathcal{I}^{u(1+\delta)} \cap S_i)$ under Q_0 and

$$Q_1(F^c) \leq \begin{cases} cuN^{-3d-1} & \text{if } k = 1, \\ cu \sum_{i \neq j} \mathcal{E}(S_i + x_i, S_j + x_j) & \text{if } k > 1, \end{cases} \quad (4.3)$$

where F is the event from (4.2). (Recall the definition of \mathcal{E} from (1.9) and the bound on it from (1.13).)

Proof. Let $c_2 = 2c_4$ where c_4 is the constant from Corollary 5.3. We first apply Corollary 5.3 with $\frac{\delta}{2}$ in place of δ (note that $1 > \frac{\delta}{2} \geq c_4 \frac{r_N}{h_N}$) to construct on a space $(\Omega_3, \mathcal{A}_3, Q_3)$ a coupling of X with law P_{q_z} and two Poisson point processes μ_1, μ_2 on $\mathcal{T}_{\bar{B}}$ with intensities $u(1 - \frac{\delta}{2})\nu, \delta u\nu$ respectively such that

$$Q_3(\forall i, \mathcal{I}(\mu_1) \cap (S_i + x_i) \subset X(0, D_{[uK_N]}) \cap (S_i + x_i) \subset (\mathcal{I}(\mu_2) \cup \mathcal{I}(\mu_1)) \cap (S_i + x_i)) \geq \frac{1}{1 - cuN^{-3d-1}}. \quad (4.4)$$

For the case $k > 1$ we will need the following lemma:

Lemma 4.2. ($N \geq c(\varepsilon)$) Let μ be a Poisson process on $\mathcal{T}_{\bar{B}}$ of intensity $s\nu, s \geq 0$. We can then define (by extending the space) an iid collection η_1, \dots, η_k of Poisson processes such that $\eta_i \stackrel{\text{law}}{=} \mu$ and

$$Q_1(\exists i, \mathcal{I}(\eta_i) \cap (S_i + x_i) \neq \mathcal{I}(\mu) \cap (S_i + x_i)) \leq 2s \sum_{i,j:j \neq i} \mathcal{E}(S_i + x_i, S_j + x_j). \quad (4.5)$$

Proof. For all j, i let $\mathcal{A}_{j,i} \subset \mathcal{T}_{\bar{B}}$ denote the set $\{H_{S_j+x_j} < H_{S_i+x_i} < T_{\bar{B}}\}$, and for all i let $\mathcal{B}_i \subset \mathcal{T}_{\bar{B}}$ denote $\cup_{j:j \neq i} \mathcal{A}_{j,i}$ and let $\mathcal{C}_i \subset \mathcal{T}_{\bar{B}}$ denote $\{H_{S_i+x_i} < T_{\bar{B}}\}$. For each i make the decomposition $\mu = \phi_i + \psi_i$ where $\phi_i = 1_{\mathcal{C}_i \cap \mathcal{B}_i^c} \mu$ and $\psi_i = 1_{\mathcal{C}_i^c \cup \mathcal{B}_i} \mu$. Because $\mathcal{C}_i \cap \mathcal{B}_i^c$ (“the excursion reaches $S_i + x_i$ first”) are disjoint the $\phi_i, i = 1, \dots, k$ are mutually independent Poisson point processes. Now extend the space by adding an independent collection of

point processes $\psi'_i, i = 1, \dots, k$ such that $\psi'_i \stackrel{\text{law}}{=} \psi_i$ and define $\eta_i = \psi'_i + \phi_i$. Then the η_i are independent and $\eta_i \stackrel{\text{law}}{=} \mu$, so to complete the proof of the lemma it suffices to show (4.5). Note that for each i :

$$\begin{aligned} Q_1(\mathcal{I}(\eta_i) \cap (S_i + x_i) \neq \mathcal{I}(\mu) \cap (S_i + x_i)) &\leq Q_1(\psi_i(\mathcal{C}_i) \neq 0 \text{ or } \psi'_i(\mathcal{C}_i) \neq 0) \\ &\stackrel{(1.32)}{\leq} 2sK_N P_q(\mathcal{B}_i) \\ &\leq 2s \sum_{j:j \neq i} K_N P_q(\mathcal{A}_{j,i}). \end{aligned} \quad (4.6)$$

If we let $K = (S_j + x_j) \cup (S_i + x_i)$ then (provided $N \geq c(\varepsilon)$ so that $K \subset \mathbb{T}_N \times (-r_N, r_N)$)

$$\begin{aligned} K_N P_q(\mathcal{A}_{j,i}) &= \sum_{x \in S_j + x_j} K_N P_q(H_K < T_{\tilde{B}}, X_{H_K} = x) P_x(H_{S_i + x_i} < T_{\tilde{B}}) \\ &\stackrel{(1.14), (1.8)}{=} \sum_{x \in S_j + x_j, y \in S_i + x_i} e_{K, \tilde{B}}(x) g_{\tilde{B}}(x, y) e_{S_i + x_i, \tilde{B}}(y) \\ &\stackrel{(1.6), (1.9)}{\leq} \mathcal{E}(S_j + x_j, S_i + x_i). \end{aligned} \quad (4.7)$$

Now combining (4.6) and (4.7) we get (4.5). \square

We now continue with the proof of Theorem 4.1. If $k > 1$ we apply Lemma 4.2 once for μ_1 and once for μ_2 and extend our space with independent $\mu_n^i, n = 1, 2, i = 1, \dots, k$, such that $\mu_1^i \stackrel{\text{law}}{=} \mu_1$ and $\mu_2^i \stackrel{\text{law}}{=} \mu_2$ for all i and

$$\begin{aligned} Q_1(F'^c) &\leq cu \sum_{i \neq j} \mathcal{E}(S_i + x_i, S_j + x_j) \text{ where } F' \text{ is the event} \\ &\{(\mathcal{I}(\mu_n^i) - x_i) \cap S_i = (\mathcal{I}(\mu_n) - x_i) \cap S_i \text{ for all } n = 1, 2, i = 1, \dots, k\}, \end{aligned} \quad (4.8)$$

If $k = 1$ then simply define $\mu_1^1 = \mu_1$ and $\mu_2^1 = \mu_2$. When $N \geq c(\varepsilon)$ we now apply Proposition 6.1 with $x = x_i$, $\mu = \mu_1^i$, $u(1 - \delta/2)$ in place of u , $u_- = u(1 - \delta)$ and $u_+ = u(1 - \frac{1}{4}\delta)$ (note that $\frac{u(1 - \delta/2)}{u_-} = 1 + \frac{\delta/2}{1 - \delta} \geq N^{-c_5}$ and $\frac{u_+}{u(1 - \delta/2)} = 1 + \frac{\delta/4}{1 - \delta/2} \geq N^{-c_5(\varepsilon)}$ for $N \geq c(\varepsilon)$) once for every $1 \leq i \leq k$, each time extending our space by adding a pair of independent random sets $\mathcal{I}_{1,i}, \mathcal{I}_{2,i}$ depending only on μ_1^i with the distribution under Q_0 of $\mathcal{I}^{u(1 - \delta)} \cap B(0, N^{1 - \varepsilon})$ and $\mathcal{I}^{u \frac{3}{4}\delta} \cap B(0, N^{1 - \varepsilon})$ respectively, such that for each i

$$Q_1(\forall i, \mathcal{I}_{1,i} \subset (\mathcal{I}(\mu_1^i) - x_i) \cap B(0, N^{1 - \varepsilon}) \subset \mathcal{I}_{1,i} \cup \mathcal{I}_{2,i}) \geq 1 - cukN^{-10(d+1)}. \quad (4.9)$$

We then apply Proposition 6.1 again, this time with $\mu = \mu_2^i$, $u\delta$ in place of u , $u_- = 0$ and $u_+ = u \frac{5\delta}{4}$ (so that $\frac{u_+}{u} = \frac{5}{4} \geq N^{-c_5(\varepsilon)}$ for $N \geq c(\varepsilon)$) once for every $1 \leq i \leq k$, each time extending our space by adding a random set $\mathcal{I}_{3,i}$ depending only on μ_2^i and distributed as $\mathcal{I}^{u \frac{5\delta}{4}} \cap B(0, N^{1 - \varepsilon})$ under Q_0 such that

$$Q_1(\forall i, (\mathcal{I}(\mu_2^i) - x_i) \cap B(0, N^{1 - \varepsilon}) \subset \mathcal{I}_{3,i}) \geq 1 - cukN^{-10(d+1)}. \quad (4.10)$$

We now define $\mathcal{I}_i^{u(1 - \delta)} \cap S_i$ and $\mathcal{I}_i^{u(1 + \delta)} \cap S_i$ by

$$\mathcal{I}_i^{u(1 - \delta)} \cap S_i = \mathcal{I}_{1,i} \cap S_i \text{ and } \mathcal{I}_i^{u(1 + \delta)} \cap S_i = (\mathcal{I}_{1,i} \cup \mathcal{I}_{2,i} \cup \mathcal{I}_{3,i}) \cap S_i, 1 \leq i \leq k.$$

Since $\mathcal{I}_{1,i}$, $\mathcal{I}_{2,i}$ and $\mathcal{I}_{3,i}$ are independent, we get from (1.36) that $(\mathcal{I}_i^{u(1-\delta)} \cap S_i, \mathcal{I}_i^{u(1+\delta)} \cap S_i)$ has the law of $(\mathcal{I}^{u(1-\delta)} \cap S_i, \mathcal{I}^{u(1+\delta)} \cap S_i)$ under Q_0 . Also the collection $((\mathcal{I}_i^{u(1-\delta)} \cap S_i, \mathcal{I}_i^{u(1+\delta)} \cap S_i))_{i=1}^k$ is independent so it only remains to show (4.3). But (4.3) in the case $k = 1$ follows directly from (4.4), (4.9) and (4.10), and if $k > 1$ it follows from (4.4), (4.8), (4.9) and (4.10) (using the crude bound $k \leq cN^{d+1}$ and (1.17)). \square

We have now completed the proofs of all the main results this article (Theorem 0.1, its corollaries and Theorem 4.1) conditionally on the results of Sections 5 and 6.

5 Coupling random walk with the Poisson process of excursions

In this section we state and prove Corollary 5.3 which couples E_N -valued random walk X with two independent Poisson processes of excursions μ_1, μ_2 , i.e. Poisson processes on $\mathcal{T}_{\tilde{B}}$ with intensity proportional to ν (see (1.32)), such that with high probability the trace $X(0, D_{[uK_N]})$ (for appropriate u) dominates the trace of μ_1 (see (1.33)) and such that the union of the traces of μ_1 and μ_2 dominate $X(0, D_{[uK_N]})$ (cf. (5.23)). The majority of the work will be to couple X with

$$\text{iid } E_N - \text{valued processes } \hat{X}^1, \hat{X}^2, \dots, \hat{X}'^1, \hat{X}'^2, \dots, \text{ with law } \kappa_q \quad (5.1)$$

(see (1.31)), such that for suitable u and δ the following double inclusion event holds with high probability:

$$\begin{aligned} I = \left\{ \bigcup_{i=1}^{[u(1-\delta)K_N]} \hat{X}^i(0, D_1) \right\} &\subset \bigcup_{i=1}^{[uK_N]} X(R_i, D_i) \\ &\subset \bigcup_{i=1}^{[u(1+\delta/2)K_N]} \hat{X}^i(0, D_1) \cup \bigcup_{i=1}^{[u\delta/2K_N]} \hat{X}'^i(0, D_1) \end{aligned} \quad (5.2)$$

To get μ_1, μ_2 one must then carry out “poissonization”, that is one must “put a Poisson number of iid the excursions” into each of μ_1 and μ_2 . This relatively simple step is carried out in Corollary 5.3.

The more challenging step of coupling X with the iid excursions $\hat{X}^1, \hat{X}^2, \dots, \hat{X}'^1, \hat{X}'^2, \dots$, is carried out in Proposition 5.1. To prove this proposition we first quote a result from [21] that couples X with “conditionally independent” excursions $\tilde{X}^1, \tilde{X}^2, \dots$ which are such that conditionally on $\tilde{X}_{D_1}^i \in \mathbb{T}_N \times \{zh_N\}$, where $z = \pm 1$, the next excursion \tilde{X}^{i+1} has law $\kappa_{q_{zr_N}}$. We then couple the conditionally independent excursions \tilde{X}^i with the truly independent excursions $\hat{X}^1, \hat{X}^2, \dots, \hat{X}'^1, \hat{X}'^2, \dots$ by using Sanov’s theorem for the empirical distribution of successive pairs of values of the Markov chain $(\frac{1}{h_N} \tilde{X}_{D_1}^i)_{i \geq 1}$ (with state space $\{-1, 1\}$) to show that for any given $z_1 \in \{-r_N, r_N\}$ and $z_2 \in \{-h_N, h_N\}$ the number of \tilde{X}^i that start in $\mathbb{T}_N \times \{z_1\}$ and end in $\mathbb{T}_N \times \{z_2\}$ is close to what this value would be if the \tilde{X}^i were truly independent.

Weaker forms of the “upper inclusions” in Proposition 5.1 and Corollary 5.3 appeared as Propositions 3.1 and 4.1 in [21]. However, as opposed to the results in this paper, the results in [21] require that u and δ are fixed as $N \rightarrow \infty$. Our proofs follows the proofs in [21] with the most important improvement taking the form of the improved bound (5.18)

on the empirical distribution of successive pairs of the Markov chain $(\frac{1}{h_N}\tilde{X}_{D_1}^i)_{i \geq 1}$, which allows for δ to go to zero as $N \rightarrow \infty$, as long as it does not do so too quickly.

Proposition 5.1. *($d \geq 2$) For any $N \geq c$ and $z \in [-N, N]$ one can construct on a space $(\Omega_2, \mathcal{A}_2, Q_2)$ a process X with law P_{q_z} and processes $\hat{X}^1, \hat{X}^2, \dots, \hat{X}'^1, \hat{X}'^2, \dots$, as in (5.1) such that for any u and δ satisfying $uK_N \geq (\log N)^6$ and $\frac{1}{2} \geq \delta \geq c_3 \frac{r_N}{h_N}$ one has, for I as in (5.2),*

$$Q_2(I^c) \leq cuN^{-3d-1}. \quad (5.3)$$

Proof. Let $X = (Y, Z)$ where Y is \mathbb{T}_N -valued and Z is \mathbb{Z} -valued. For $\gamma = (z_1, z_2) \in \Gamma \stackrel{\text{def}}{=} \{-r_N, r_N\} \times \{-h_N, h_N\}$ we let P_γ denote the law of $X_{\cdot \wedge D_1}$ under $P_{q_{z_1}}$ conditioned on $\{Z_{D_1} = z_2\}$. By Proposition 2.2 of [21] we can construct a coupling $(Q', \Omega', \mathcal{A}')$ of the random walk X with law P_{q_z} , a \mathbb{Z}^2 -valued process $(\tilde{Z}_{R,k}, \tilde{Z}_{D,k})_{k \geq 1}$ distributed as $(Z_{R,k}, Z_{D,k})_{k \geq 1}$ under P_{q_z} and E_N -valued processes $(\tilde{X}^k)_{k \geq 1}$ which conditionally on $(\tilde{Z}_{R,k}, \tilde{Z}_{D,k})_{k \geq 1}$ are independent with the law of \tilde{X}^k given by $P_{\tilde{Z}_{R,k}, \tilde{Z}_{D,k}}$, such that $Q'(X_{(R_k + \cdot) \wedge D_k} \neq \tilde{X}^k) \leq cN^{-4d}$ for all k . Thus:

$$Q'(\exists k \leq 2uK_N \text{ such that } X(R_k, D_k) \neq \tilde{X}^k(R_1, D_1)) \stackrel{(1.16)}{\leq} cuN^{-3d-1}. \quad (5.4)$$

We will construct on a space (Σ, \mathcal{B}, M) a coupling of a sequence of processes $(\bar{X}^k)_{k \geq 1}$ with the law of $(\tilde{X}^k)_{k \geq 1}$ under Q' and $\hat{X}^1, \hat{X}^2, \dots, \hat{X}'^1, \hat{X}'^2, \dots$, iid with law κ_q , such that:

$$M(F'^c) \leq \exp(-c(\log N)^2) \stackrel{u \geq N^{-1000d}, N \geq c}{\leq} cuN^{-3d-1}. \quad (5.5)$$

where F' is the event given in (5.2) with $\bigcup_{i=1}^{[uK_N]} X(R_i, D_i)$ replaced by $\bigcup_{i=1}^{[uK_N]} \bar{X}^i(R_1, D_1)$. Using the argument below (3.22) in [21], this, together with (5.4), is enough to show the existence of the desired coupling of X and \hat{X}^i, \hat{X}'^i such that (5.3) holds (essentially speaking because we can construct $(\Omega', \mathcal{A}', Q')$ such that the regular conditional probability of X given $(\tilde{X}^i)_{i \geq 1}$ exists).

We thus proceed with the construction of (Σ, \mathcal{B}, M) . We start by defining on (Σ, \mathcal{B}, M) the following collections of processes

$$\gamma_k \in \mathbb{Z}^2, k \geq 1, (\gamma_k \in \Gamma \text{ if } k \geq 2) \text{ with the law of } (Z_{R_k}, Z_{D_k})_{k \geq 1} \text{ under } P_{q_z}, \quad (5.6)$$

$$\gamma'_k \in \Gamma, k \geq 1, \text{ iid, where } \gamma'_k \text{ has the law of } (Z_{R_1}, Z_{D_1}) \text{ under } P_q, \quad (5.7)$$

$$\text{for all } \gamma \in \Gamma \text{ an iid sequence } (\zeta_i^\gamma(\cdot))_{i \geq 1} \text{ of processes with law } P_\gamma, \quad (5.8)$$

$$\text{an iid sequence } (\hat{X}'^i)_{i \geq 1} \text{ of processes with law } \kappa_q, \quad (5.9)$$

such that the collections are mutually independent. Also define for every $\gamma \in \Gamma$:

$$N_k(\gamma) = |\{j \in [2, k] : \gamma_j = \gamma\}|, k \geq 2, N'_k(\gamma) = |\{j \in [1, k] : \gamma'_j = \gamma\}|, k \geq 1. \quad (5.10)$$

We further let:

$$\begin{aligned} \hat{X}^k &= \zeta_{N'_k(\gamma'_k)}^{\gamma'_k}(\cdot) \text{ for } k \geq 1, \\ \bar{X}^k &= \zeta_{N_k(\gamma_k)}^{\gamma_k}(\cdot) \text{ for } k \geq 2 \text{ and } \bar{X}^1 = \hat{X}'^{i_0}_{H_{\mathbb{T}_N \times \{z\} + \cdot}}, \end{aligned} \quad (5.11)$$

where $i_0 = \inf\{i \geq 1 : J_i \text{ holds}\}$, $J_i = \{\hat{X}'^i \text{ hits } \mathbb{T}_N \times \{z\} \text{ before leaving } \tilde{B}, \hat{X}'^i_{D_1} \in \mathbb{T}_N \times \{z_2\}\}$ and $\gamma_1 = (z, z_2)$. We then have that:

$$(\bar{X}^k)_{k \geq 1} \text{ under } M \text{ has the same law as } (\tilde{X}^k)_{k \geq 1} \text{ under } Q', \text{ and} \quad (5.12)$$

$$\hat{X}^1, \hat{X}^2, \dots, \hat{X}'^1, \hat{X}'^2, \dots, \text{ under } M \text{ are iid with law } \kappa_q. \quad (5.13)$$

Thus it only remains to show (5.5). We introduce the “good event”:

$$\mathcal{G} = \{i_0 \leq \left\lceil \frac{u\delta}{2} K_N \right\rceil, N'_{[u(1-\delta)K_N]}(\gamma) \leq N_{[uK_N]}(\gamma) \leq N'_{[u(1+\frac{\delta}{2})K_N]}(\gamma) \forall \gamma \in \Gamma\}.$$

By (5.10) and (5.11) we have $\mathcal{G} \subset F'$ so to show (5.5) it suffices to show that

$$M(\mathcal{G}^c) \leq c \exp(-c(\log N)^2). \quad (5.14)$$

Note that $M(i_0 > n) \leq M(J_1^c)^n, n = 0, 1, \dots$, and $M(J_1) \geq \frac{49}{100}$ for $N \geq c$ by a one dimensional random walk calculation (see (3.23) of [21]). So if $N \geq c$ and $\delta \geq \frac{r_N}{h_N}$:

$$M(i_0 > \left\lceil \frac{u\delta}{2} K_N \right\rceil) \leq \left(\frac{51}{100} \right)^{\left\lceil \frac{u\delta}{2} K_N \right\rceil} \leq c \exp \left(-cuK_N \frac{r_N}{h_N} \right). \quad (5.15)$$

Recall that the sequence $\gamma'_1, \gamma'_2, \dots$, is iid and note that $M(\gamma'_1 = \gamma) = \frac{1}{2}p_N 1_{\{z_1 z_2 > 0\}} + \frac{1}{2}q_N 1_{\{z_1 z_2 < 0\}}$, for $\gamma = (z_1, z_2) \in \Gamma$, where $p_N = \frac{1}{2} + \frac{1}{2} \frac{r_N}{h_N}$ and $q_N = \frac{1}{2} - \frac{1}{2} \frac{r_N}{h_N}$. Using the exponential Chebyshev inequality we get that if $N \geq c$ and $\delta \geq 6 \frac{r_N}{h_N}$ (ensuring that $\frac{1}{2}q_N [u(1 + \frac{\delta}{2})K_N]$, the “typical size” of $N'_{[u(1+\frac{\delta}{2})K_N]}(z_1, z_2)$ when $z_1 z_2 < 0$, is “much larger” than $\frac{1}{4}u(1 + \frac{\delta}{4})K_N$):

$$\begin{aligned} \sup_{\gamma \in \Gamma} M \left(N'_{[u(1-\delta)K_N]}(\gamma) \geq \frac{1}{4}u(1 - \frac{\delta}{4})K_N \right) &\leq c \exp \left(-cuK_N \left(\frac{r_N}{h_N} \right)^2 \right), \\ \sup_{\gamma \in \Gamma} M \left(N'_{[u(1+\frac{\delta}{2})K_N]}(\gamma) \leq \frac{1}{4}u(1 + \frac{\delta}{4})K_N \right) &\leq c \exp \left(-cuK_N \left(\frac{r_N}{h_N} \right)^2 \right). \end{aligned} \quad (5.16)$$

If we write $\gamma^i = (z_1^i, z_2^i)$ then $V_i = \frac{z_2^i}{h_N}, i \geq 1$, is a Markov chain on $\{-1, 1\}$ with transition probabilities $P(V_{i+1} = a | V_i = b) = p_N 1_{\{ab > 0\}} + q_N 1_{\{ab < 0\}}$ for $a, b = \pm 1$. Also $\gamma^i = (V_{i-1}r_N, V_i h_N)$ almost surely for all $i \geq 2$. The sequence of consecutive pairs $(V_{i-1}, V_i), i \geq 2$, is a Markov chain on $\{-1, 1\}^2$. Let $(U_i)_{i \geq 0}$ under the probability \tilde{R}_σ be a Markov chain on $\{-1, 1\}^2$ with the same transition probabilities as (V_{i-1}, V_i) but with $U_0 = \sigma \in \{-1, 1\}^2$ almost surely. If $a, b = \pm 1$ let I_1 and I_2 denote the events $\left\{ \sum_{i=1}^{[uK_N]-1} 1_{\{U_i=(a,b)\}} \geq \frac{1}{4}u(1 + \frac{\delta}{4})K_N \right\}$ and $\left\{ \sum_{i=1}^{[uK_N]-1} 1_{\{U_i=(a,b)\}} \leq \frac{1}{4}u(1 - \frac{\delta}{4})K_N \right\}$ respectively. Then by (5.10) we have:

$$\begin{aligned} M \left(N_{[uK_N]}((ar_N, bh_N)) \geq \frac{1}{4}u(1 + \frac{\delta}{4})K_N \right) &\leq \sup_\sigma \tilde{R}_\sigma(I_1), \\ M \left(N_{[uK_N]}((ar_N, bh_N)) \leq \frac{1}{4}u(1 - \frac{\delta}{4})K_N \right) &\leq \sup_\sigma \tilde{R}_\sigma(I_2). \end{aligned} \quad (5.17)$$

We have the following lemma:

Lemma 5.2. ($N \geq c$) If $\frac{1}{2} \geq \delta \geq 32 \frac{r_N}{h_N}$ then for all $a, b = \pm 1$

$$\sup_{\sigma} \tilde{R}_{\sigma}(I_i) \leq \exp \left(-cuK_N \left(\frac{r_N}{h_N} \right)^2 \right) \text{ for } i = 1, 2. \quad (5.18)$$

Proof. By symmetry it suffices to check the cases $a = 1, b = 1$ and $a = 1, b = -1$. For a probability $\mu(\cdot, \cdot)$ on $\{-1, 1\}^2$ we write μ_1, μ_2 for its marginals and $\mu(j|i) = \frac{\mu(i,j)}{\mu_1(i)}$. By Theorem 3.1.13 p. 79 of [10] and by sub-additivity (cf. Lemma 6.1.11 p. 255 and Lemma 6.3.1 p. 273 of [10]), we have that for all $n \geq 1$ and $x \in (0, \frac{1}{4}]$

$$\begin{aligned} \inf_{\sigma} \tilde{R}_{\sigma}(\frac{1}{n} \sum_{i=0}^{n-1} 1_{\{U_i=(1,b)\}} \geq \frac{1}{4} + x) &\leq \exp(-n\Psi_{2,N}^+(x)), \\ \inf_{\sigma} \tilde{R}_{\sigma}(\frac{1}{n} \sum_{i=1}^n 1_{\{U_i=(1,b)\}} \leq \frac{1}{4} - x) &\leq \exp(-n\Psi_{2,N}^-(x)) \end{aligned} \quad (5.19)$$

where

$$\begin{aligned} \Psi_{2,N}^+(x) &= \inf \{ H_{2,N}(\mu) : \mu \text{ probability on } \{-1, 1\}^2, \mu_1 = \mu_2, \mu(1, b) \geq \frac{1}{4} + x \}, \\ \Psi_{2,N}^-(x) &= \inf \{ H_{2,N}(\mu) : \mu \text{ probability on } \{-1, 1\}^2, \mu_1 = \mu_2, \mu(1, b) \leq \frac{1}{4} - x \}, \\ H_{2,N}(\mu) &= \mu_1(1) \left\{ \mu(1|1) \log \frac{\mu(1|1)}{p_N} + \mu_1(-1|1) \log \frac{\mu(-1|1)}{q_N} \right\} + \\ &\quad \mu_1(-1) \left\{ \mu(1|-1) \log \frac{\mu(1|-1)}{q_N} + \mu_1(-1|-1) \log \frac{\mu(-1|-1)}{p_N} \right\}. \end{aligned}$$

Because $\inf_{\sigma,j} \tilde{R}_{\sigma}(U_2 = j) \geq c$ we have

$$\begin{aligned} &\sup_{\sigma} \tilde{R}_{\sigma} \left(\sum_{i=1}^{[uK_N]-1} 1_{\{U_i=(1,b)\}} \geq \frac{1}{4}u(1 + \frac{\delta}{4})K_N \right) \\ &\leq \frac{1}{c} \inf_{\sigma} \tilde{R}_{\sigma} \left(\sum_{i=1}^{[uK_N]+1} 1_{\{U_i=(1,b)\}} \geq \frac{1}{4}u(1 + \frac{\delta}{4})K_N \right) \\ &\stackrel{uK_N \delta \geq 2, (5.19)}{\leq} c \exp \left(-([uK_N] + 1) \Psi_{2,N}^+(\frac{\delta}{32}) \right) \leq c \exp \left(-uK_N \Psi_{2,N}^+(\frac{\delta}{32}) \right), \end{aligned} \quad (5.20)$$

and similarly

$$\sup_{\sigma} \tilde{R}_{\sigma} \left(\sum_{i=1}^{[uK_N]-1} 1_{\{U_i=(1,b)\}} \leq \frac{1}{4}u(1 - \frac{\delta}{4})K_N \right) \leq c \exp \left(-uK_N \Psi_{2,N}^-(\frac{\delta}{32}) \right). \quad (5.21)$$

To conclude the proof of the lemma it thus suffices to show that for $b = -1, 1$:

$$\Psi_{2,N}^{\pm} \left(\frac{r_N}{h_N} \right) \geq c \left(\frac{r_N}{h_N} \right)^2. \quad (5.22)$$

Consider the function $f_p(x) = x \log \frac{x}{p} + (1-x) \log \frac{1-x}{1-p}$ for $p \in (0, 1)$ and $x \in [0, 1]$. Since $f_p''(x) = \frac{1}{x} + \frac{1}{1-x} \geq 4$ and $f_p(p) = f_p'(p) = 0$ integrating twice gives that $f_p(x) \geq 2(x-p)^2$ for all $p \in (0, 1)$ and $x \in [0, 1]$. Using this and $\mu_1(1), \mu_1(-1) \geq 1$ we get

$$\begin{aligned} H_{2,N}(\mu) &\geq 2 \{ \mu_1(1)^2 (\mu(1|1) - p_N)^2 + \mu_1(-1)^2 (\mu(-1|-1) - p_N)^2 \} \\ &= 2 \{ (\mu(1, 1)(1 - p_N) - p_N \mu(1, -1))^2 + (\mu(-1, -1)(1 - p_N) - p_N \mu(-1, 1))^2 \} \end{aligned}$$

Write $\theta = \frac{1}{2}p_N$, $\mu(1, 1) = \theta + y$ and $\mu(-1, -1) = \theta + z$. If $\mu_1 = \mu_2$ then $\mu(1, -1) = \mu(-1, 1) = \frac{1-y-z}{2} - \theta$ and plugging this into the above formula we get

$$\begin{aligned} H_{2,N}(\mu) &\geq 2 \left\{ (y(1-\theta) + z\theta)^2 + (z(1-\theta) + y\theta)^2 \right\} \\ &\stackrel{yz \geq -\frac{y^2+z^2}{2}}{\geq} 2(1-2\theta)^2 \{y^2 + z^2\} \\ &\geq c \left\{ (\mu(1, 1) - \frac{1}{2}p_N)^2 + (\mu(-1, -1) - \frac{1}{2}p_N)^2 \right\}. \end{aligned}$$

Now if $|\mu(1, 1) - \frac{1}{4}| \geq \frac{r_N}{h_N}$ or $|\mu(-1, -1) - \frac{1}{4}| \geq \frac{r_N}{h_N}$ and $\mu_1 = \mu_2$ then $|\mu(1, 1) - \frac{1}{2}p_N| \geq \frac{3}{4}\frac{r_N}{h_N}$ or $|\mu(-1, -1) - \frac{1}{2}p_N| \geq \frac{3}{4}\frac{r_N}{h_N}$, so by the above inequality $H_{2,N}(\mu) \geq c \left(\frac{r_N}{h_N}\right)^2$ and thus (5.22) holds when $b = 1$. Furthermore if $|\mu(1, -1) - \frac{1}{4}| \geq \frac{r_N}{h_N}$ and $\mu_1 = \mu_2$ then $|\mu(1, 1) - \frac{1}{4}| \geq \frac{r_N}{h_N}$ or $|\mu(-1, -1) - \frac{1}{4}| \geq \frac{r_N}{h_N}$ so (5.22) holds also when $b = -1$. Thus the proof of the lemma is complete. \square

We can now finish the proof of Proposition 5.1. Combining (5.15), (5.16), (5.17) and (5.18), and using the bounds $c \exp(-cuK_N \frac{r_N}{h_N}) \leq c \exp(-cuK_N (\frac{r_N}{h_N})^2) \leq c \exp(-c(\log N)^2)$ (recall $uK_N \geq (\log N)^6$), we deduce that (5.14) holds. Thus the proof of Proposition 5.1 is complete. \square

We are now ready to carry out the process of “poissonization” to construct Poisson processes of excursions (i.e. Poisson processes on $\mathcal{T}_{\bar{B}}$ of intensity a multiple of ν , cf. (1.32)) from the iid excursions of the previous proposition.

Corollary 5.3. *For all $N \geq c$, $z \in [-N, N]$, $1 > \delta \geq c_4 \frac{r_N}{h_N}$ and u satisfying $uK_N \geq (\log N)^6$ we can define on a space $(\Omega_3, \mathcal{A}_3, Q_3)$ a process X with law P_{q_z} and two independent Poisson point processes μ_1, μ_2 on $\mathcal{T}_{\bar{B}}$ with intensities $u(1-\delta)\nu$ and $2\delta u\nu$ respectively such that*

$$Q_3(\mathcal{I}(\mu_1) \subset \cup_{i=1}^{\lfloor uK_N \rfloor} X(R_i, D_i) \subset \mathcal{I}(\mu_1) \cup \mathcal{I}(\mu_2)) \geq 1 - cuN^{-3d-1}. \quad (5.23)$$

Proof. Let $c_4 = 2c_3$ so that we can apply Proposition 5.1 with $\frac{\delta}{2}$ in place of δ to get a space $(\Omega_2, \mathcal{A}_2, Q_2)$ with a process X with law P_{q_z} and processes $(\hat{X}^k)_{k \geq 1}, (\hat{X}'^k)_{k \geq 1}$, as in (5.1) such that $Q_2(I'^c) \leq cuN^{-3d-1}$, where I' is the event in (5.2) with δ replaced by $\frac{\delta}{2}$. We define $(\Omega_3, \mathcal{A}_3, Q_3)$ by extending $(\Omega_2, \mathcal{A}_2, Q_2)$ with independent Poisson random variables J_1 with parameter $K_N u(1-\delta)$, J_2 with parameter $K_N u \frac{3\delta}{2}$ and J_3 with parameter $K_N u \frac{\delta}{2}$, which are also independent from $(\hat{X}^k)_{k \geq 1}, (\hat{X}'^k)_{k \geq 1}$. We then define

$$\mu_1 = \sum_{1 \leq k \leq J_1} \delta_{\hat{X}^k} \text{ and } \mu_2 = \sum_{J_1+1 \leq k \leq J_1+J_2} \delta_{\hat{X}^k} + \sum_{1 \leq k \leq J_3} \delta_{\hat{X}'^k}. \quad (5.24)$$

Then μ_1 and μ_2 are independent Poisson point processes with intensities $u(1-\delta)\nu$ and $2\delta u\nu$. It thus only remains to show (5.23). Note that the complement of the event in the left-hand side of (5.23) is included in

$$\{J_1 > [u(1-\frac{\delta}{2})K_N]\} \cup \{J_1 + J_2 < [u(1+\frac{\delta}{4})K_N]\} \cup \{J_3 < [u\frac{\delta}{4}K_N]\} \cup I'^c.$$

But using standard large deviation bounds we see that the probabilities of the first three events in the union are bounded above by $\exp(-cuK_N\delta^2) \stackrel{uK_N\delta^2 \geq (\log N)^2, N \geq c}{\leq} cuN^{-3d-1}$ and thus (5.23) follows since we already know $Q_3(I^c) \leq cuN^{-3d-1}$. \square

In finishing the proof of Corollary 5.3 we have now proved the first of the two main ingredients that were used to prove Theorem 4.1.

6 Coupling the Poisson process of excursions with random interlacements

In this section we state and prove Proposition 6.1, which couples the Poisson process of excursions with random interlacements and whose application was an important part of the proof of Theorem 4.1. It states that if we have a Poisson process of excursions μ (i.e. Poisson process on $\mathcal{T}_{\tilde{B}}$ of intensity $u\nu$ for $u \geq 0$, see (1.32)) then for any $x \in \mathbb{T}_N \times [-\frac{N}{2}, \frac{N}{2}]$ and $\varepsilon \in (0, 1)$ we can, provided N is large enough and $u_- < u$ and $u_+ > u$ are “sufficiently far” from u , construct independent random sets $\mathcal{I}_1, \mathcal{I}_2 \subset \mathbf{A} = B(0, N^{1-\varepsilon})$ such that

$$\mathcal{I}_1 \text{ has the law of } \mathcal{I}^{u_-} \cap \mathbf{A} \text{ under } Q_0 \text{ and } \mathcal{I}_2 \text{ has the law of } \mathcal{I}^{u_+ - u_-} \cap \mathbf{A} \text{ under } Q_0 \quad (6.1)$$

and with high probability $\mathcal{I}_1 \subset (\mathcal{I}(\mu) - x) \cap \mathbf{A} \subset \mathcal{I}_1 \cup \mathcal{I}_2$ (recall that \mathcal{I}^u under Q_0 is a random interlacement and that $(\mathcal{I}_1, \mathcal{I}_1 \cup \mathcal{I}_2) \stackrel{\text{law}}{=} (\mathcal{I}^{u_-} \cap \mathbf{A}, \mathcal{I}^{u_+} \cap \mathbf{A})$ by (1.36)). More precisely:

Proposition 6.1. *Assume $\varepsilon \in (0, 1)$, $N \geq c(\varepsilon)$, $x \in \mathbb{T}_N \times [-\frac{N}{2}, \frac{N}{2}]$ and let $\mathbf{A} = B(0, N^{1-\varepsilon})$. Suppose that we have a Poisson process μ on $\mathcal{T}_{\tilde{B}}$ with intensity $u\nu$, $u \geq 0$ defined on a space (Ω, \mathcal{A}, Q) . Then if $0 \leq u_- < u < u_+$, $\frac{u_+}{u}, \frac{u}{u_-} \geq N^{-c_5}$, where $c_5 = c_5(\varepsilon) > 0$, we can define a space $(\Omega', \mathcal{A}', Q')$ and random sets $\mathcal{I}_1, \mathcal{I}_2 \subset \mathbf{A}$ on the product space $(\Omega \times \Omega', \mathcal{A} \otimes \mathcal{A}', Q \otimes Q')$ such that (6.1) holds,*

$$\mathcal{I}_1, \mathcal{I}_2, \text{ are independent, } \sigma(\mu) \otimes \mathcal{A}' - \text{measurable and} \quad (6.2)$$

$$Q \otimes Q'(\mathcal{I}_1 \subset (\mathcal{I}(\mu) - x) \cap \mathbf{A} \subset \mathcal{I}_1 \cup \mathcal{I}_2) \geq 1 - cu_+ N^{-10(d+1)}. \quad (6.3)$$

Before starting the proof of Proposition 6.1 we make some definitions and state Proposition 6.2, all of which we will need in the proof. We define the box

$$A' = B(x, N^{1-\varepsilon}). \quad (6.4)$$

The first step in the proof of Proposition 6.1 will be to extract from μ a Poisson process μ' , by keeping only trajectories in μ that hit A' (the others are irrelevant for the coupling). We will see that what is left, i.e. μ' , is a Poisson process on $\mathcal{T}_{\tilde{B}}$ of intensity $u\kappa_{e_{A'}, \tilde{B}}$. We define the boxes

$$B = B(0, N^{1-\varepsilon/2}), C = B(0, \frac{N}{4}), B' = B(x, N^{1-\varepsilon/2}), C' = B(x, \frac{N}{4}). \quad (6.5)$$

Note that for $N \geq c(\varepsilon)$

$$\mathbf{A} \subset \mathbf{B} \subset \mathbf{C} \text{ and } \mathbf{A}' \subset \mathbf{B}' \subset \mathbf{C}'. \quad (6.6)$$

(One should not confuse \mathbf{B} with B from (1.2).) We further define for $k \geq 1$ the successive returns \tilde{R}_k to \mathbf{A} and departures \tilde{D}_k from \mathbf{B} , and returns \tilde{R}'_k to \mathbf{A}' and departures \tilde{D}'_k from \mathbf{B}' , analogously to (1.3). For $1 \leq l < \infty$ we then introduce the maps ϕ_l, ϕ'_l from $\{\tilde{D}_k < \infty = \tilde{R}_{k+1}\} \subset W$ and $\{\tilde{D}'_k < T_{\tilde{B}} < \tilde{R}'_{k+1}\} \subset \mathcal{T}_{\tilde{B}}$ respectively into $W^{\times l}$:

$$\phi_l(w) \stackrel{\text{def}}{=} (w((\tilde{R}_i + \cdot) \wedge \tilde{D}_i))_{1 \leq i \leq l}, \phi'_l(w) \stackrel{\text{def}}{=} (w((\tilde{R}'_i + \cdot) \wedge \tilde{D}'_i) - x)_{1 \leq i \leq l}. \quad (6.7)$$

For $1 \leq l < \infty$ we will then consider μ_l , the image of μ' under ϕ'_l , that is from each trajectory we will only keep its excursions between \mathbf{A}' and $\partial_e \mathbf{B}'$. Essentially speaking we will then be left with Poisson point processes $\mu_l, l \geq 1$, on the spaces $W^{\times l}, l \geq 1$, of intensities $u\xi_E^l, l \geq 1$, where for $w_1, \dots, w_l \in W$

$$\xi_E^l(w_1, \dots, w_l) \stackrel{\text{def}}{=} \left(\phi'_l \circ (1_{\{\tilde{D}'_i < T_{\tilde{B}} < \tilde{R}'_{i+1}\}} \kappa_{e_{\mathbf{A}', \tilde{B}}}) \right) (w_1 + x, \dots, w_l + x). \quad (6.8)$$

Recall from (1.35) that $\mathcal{I}^s \cap \mathbf{A}$ has the law of $\mathcal{I}(\mu_{\mathbf{A}, s}) \cap \mathbf{A}$ for any $s \geq 0$. It will turn out that if we consider the image of $1_{\{\tilde{D}_l < \infty = \tilde{R}_{l+1}\}} \mu_{\mathbf{A}, s}$ under ϕ_l , thereby only keeping the excursions between \mathbf{A} and $\partial_e \mathbf{B}$, we get Poisson processes *on the same spaces* $W^{\times l}, l \geq 1$, but with intensities $s\xi_{\mathbb{Z}^{d+1}}^l$, where for $w_1, \dots, w_l \in W$

$$\xi_{\mathbb{Z}^{d+1}}^l(w_1, \dots, w_l) \stackrel{\text{def}}{=} \left(\phi_l \circ (1_{\{\tilde{D}_l < \infty = \tilde{R}_{l+1}\}} P_{e_{\mathbf{A}}}^{\mathbb{Z}^{d+1}}) \right) (w_1, \dots, w_l). \quad (6.9)$$

The following comparison of $\xi_{\mathbb{Z}^{d+1}}^l$ and ξ_E^l is crucial in the proof of Proposition 6.1:

Proposition 6.2. *($N \geq c(\varepsilon)$) For all $l \geq 1$*

$$(1 - c(l)N^{-c(\varepsilon)})\xi_{\mathbb{Z}^{d+1}}^l \leq \xi_E^l \leq (1 + c(l)N^{-c(\varepsilon)})\xi_{\mathbb{Z}^{d+1}}^l. \quad (6.10)$$

We postpone the proof of Proposition 6.2 until after the proof of Proposition 6.1. In Proposition 6.1 we will use Proposition 6.2 to “thin” $\mu_l, l = 1, \dots, r$, where r is a constant, to get Poisson processes $\mu_l^-, l = 1, \dots, r$ with intensity $u_- \xi_{\mathbb{Z}^{d+1}}^l$ and “thicken” $\mu_l - \mu_l^-$ to get Poisson processes $\mu_l^+, l = 1, \dots, r$ with intensity $u_+ \xi_{\mathbb{Z}^{d+1}}^l$, such for each $l = 1, \dots, r$ we have $\mu_l^- \leq \mu_l \leq \mu_l^- + \mu_l^+$. We will then essentially speaking define the set \mathcal{I}_1 in terms of the traces of the μ_l^- and define \mathcal{I}_2 in terms of the traces of the μ_l^+ . We will pick r (see (6.15)) such that with high probability $\sum_{l > r} \mu_l = 0$. This together with the relation $\mu_l^- \leq \mu_l \leq \mu_l^- + \mu_l^+$ will allow us to prove that the inclusion in (6.3) holds with high probability. Also the relation between $\xi_{\mathbb{Z}^{d+1}}^l, \mu_{\mathbf{A}, s}$ and $\mathcal{I}^s \cap \mathbf{A}$ described above will allow us to show (6.1). We now start the proof.

Proof of Proposition 6.1. We start with by constructing the processes μ_l in the following lemma:

Lemma 6.3. *We can define on (Ω, \mathcal{A}, Q) processes $\mu_l, 1 \leq l < \infty$, such that*

$$\mu_l \text{ are independent } \sigma(\mu) - \text{measurable Poisson point processes}, \quad (6.11)$$

$$\mu_l \text{ has statespace } W^{\times l} \text{ and intensity } u\xi_E^l, \quad (6.12)$$

$$(\mathcal{I}(\mu) - x) \cap \mathbf{A} = \cup_{l \geq 1} \mathcal{I}(\mu_l) \text{ almost surely.} \quad (6.13)$$

Proof. Define on (Ω, \mathcal{A}, Q) the processes $\mu' = \sum_{n \geq 0} 1_{\{w_n \text{ hits } \mathcal{A}'\}} \delta_{w_n(H_{\mathcal{A}'} + \cdot)}$ when $\mu = \sum_{n \geq 0} \delta_{w_n}$. Then μ' is a Poisson process on $\mathcal{T}_{\tilde{B}}$ of intensity

$$uK_N P_q(X_{(H_{\mathcal{A}'} + \cdot) \wedge T_{\tilde{B}}} \in dw) \stackrel{(1.15)}{=} u\kappa_{e_{\mathcal{A}', \tilde{B}}}(dw)$$

and

$$(\mathcal{I}(\mu) - x) \cap \mathbf{A} = (\mathcal{I}(\mu') - x) \cap \mathbf{A} \text{ almost surely.} \quad (6.14)$$

Furthermore define for $1 \leq l < \infty$ the process μ_l as the image of $1_{\{\tilde{D}'_l < T_{\tilde{B}} < \tilde{R}'_{l+1}\}} \mu'$ under ϕ_l . Then since $\{\tilde{D}'_l < T_{\tilde{B}} < \tilde{R}'_{l+1}\}$ are disjoint, and μ' only depends on μ we get (6.11). By (6.7) and (6.8) we get (6.12). Finally (6.13) follows by (6.14) and (6.7) and since $\cup_{l \geq 1} \{\tilde{D}'_l < T_{\tilde{B}} < \tilde{R}'_{l+1}\}$ equals the support of μ' . \square

The next step in the proof of Proposition 6.1 is to construct the processes μ_l^-, μ_l^+ . First let us define

$$r = \left\lceil \frac{10(d+1)}{c_6(\varepsilon)} \right\rceil + 1, \quad (6.15)$$

where $c_6 = c_6(\varepsilon)$ is the constant from Lemma 6.6. We have the following lemma:

Lemma 6.4. ($N \geq c(\varepsilon), 0 \leq u_- < u < u_+, \frac{u_+}{u}, \frac{u_-}{u} \geq N^{-c_5}$) *We can construct a space $(\Omega', \mathcal{A}', Q')$ and processes $\mu_l^-, \mu_l^+, 1 \leq l \leq r$ such that*

$$\mu_l^-, \mu_l^+, 1 \leq l \leq r, \text{ are independent } \sigma(\mu) \times \mathcal{A}'\text{-measurable,} \quad (6.16)$$

$$\mu_l^- \text{ is a Poisson point process on } W^{\times l} \text{ of intensity } u_- \xi_{\mathbb{Z}^{d+1}}^l, \quad (6.17)$$

$$\mu_l^+ \text{ is a Poisson point process on } W^{\times l} \text{ of intensity } (u_+ - u_-) \xi_{\mathbb{Z}^{d+1}}^l, \quad (6.18)$$

$$\mu_l^- \leq \mu_l \leq \mu_l^- + \mu_l^+ \text{ almost surely.} \quad (6.19)$$

Proof. By (6.10) it follows (if we define $c_5 = c_5(\varepsilon)$ appropriately) that for $N \geq c(\varepsilon)$

$$u_- \xi_{\mathbb{Z}^{d+1}}^l \leq u \xi_E^l \leq u_+ \xi_{\mathbb{Z}^{d+1}}^l, \text{ for } 1 \leq l \leq r. \quad (6.20)$$

Since μ_l has intensity $u \xi_E^l$ and $u_- \xi_{\mathbb{Z}^{d+1}}^l \leq u \xi_E^l$ we can thin μ_l (by defining the appropriate random variables on $(\Omega', \mathcal{A}', Q')$) to get μ_l^- such that $\mu_l^-, \mu_l - \mu_l^-$ are independent, $\sigma(\mu) \times \mathcal{A}'$ -measurable, $\mu_l^- \leq \mu_l$ and (6.18) holds. The $\mu_l - \mu_l^-$ then have intensity $u \xi_E^l - u_- \xi_{\mathbb{Z}^{d+1}}^l \stackrel{(6.20)}{\leq} (u_+ - u_-) \xi_{\mathbb{Z}^{d+1}}^l$ so we can (by extending the space $(\Omega', \mathcal{A}', Q')$) thicken them to get μ_l^+ such that (6.16), (6.18) and (6.19) hold. \square

We now continue the proof of Proposition 6.1 by further extending $(\Omega', \mathcal{A}', Q')$ with two independent Poisson point process $\bar{\mu}^+$ and $\bar{\mu}^-$ on W of respective intensities $u_- 1_{\{\tilde{D}_{r+1} < \infty\}} P_{e_A}^{\mathbb{Z}^{d+1}}$ and $(u_+ - u_-) 1_{\{\tilde{D}_{r+1} < \infty\}} P_{e_A}^{\mathbb{Z}^{d+1}}$ respectively and define on $(\Omega \times \Omega', \mathcal{A} \otimes \mathcal{A}', Q \otimes Q')$

$$\mathcal{I}_1 = (\cup_{l=1}^r \mathcal{I}(\mu_l^+) \cup \mathcal{I}(\bar{\mu}^+)) \cap \mathbf{A}, \mathcal{I}_2 = (\cup_{l=1}^r \mathcal{I}(\mu_l^-) \cup \mathcal{I}(\bar{\mu}^-)) \cap \mathbf{A}. \quad (6.21)$$

Then (6.2) holds by (6.16) and the definitions of $\bar{\mu}^+$ and $\bar{\mu}^-$. We check (6.1) in the following lemma:

Lemma 6.5. (6.1) holds for $\mathcal{I}_1, \mathcal{I}_2$ as in (6.21).

Proof. Recall the definition of μ_{A, u_-} from (1.35). Similarly to (6.14) we have:

$$\mathcal{I}(\mu_{A, u_-}) \cap A = \left(\bigcup_{l=1}^r \mathcal{I}(\phi_l(1_{\{\tilde{D}_l < \infty = \tilde{R}_{l+1}\}} \mu_{A, u_-})) \cup \mathcal{I}(1_{\{\tilde{D}_{r+1} < \infty\}} \mu_{A, u_-}) \right) \cap A. \quad (6.22)$$

Also similarly to (6.11) and (6.12) the $\phi_l(1_{\{\tilde{D}_l < \infty = \tilde{R}_{l+1}\}} \mu_{A, u_-})$, $1 \leq l \leq r$, $1_{\{\tilde{D}_{r+1} < \infty\}} \mu_{A, u_-}$, are independent Poisson point processes and have intensities $u_- \xi_{\mathbb{Z}^{d+1}}^l$, $1 \leq l \leq r$, and $u_- 1_{\{\tilde{D}_{r+1} < \infty\}} P_{e_A}^{\mathbb{Z}^{d+1}}$ respectively. These coincide with the intensities of μ_l^- , $1 \leq l \leq r$, $\bar{\mu}_l^-$, so from (6.21) and (6.22) we see that $\mathcal{I}_1 \stackrel{\text{law}}{=} \mathcal{I}(\mu_{A, u_-}) \cap A \stackrel{\text{law}, (1.35)}{=} \mathcal{I}^{u_-} \cap A$. Similarly $\mathcal{I}_2 \stackrel{\text{law}}{=} \mathcal{I}^{u_+ - u_-} \cap A$ so the proof of the lemma is complete. \square

Continuing with the proof of Proposition 6.1 we see that we are done once we have shown (6.3). We have (noting that by (6.12) the process $\sum_{l>r} \mu_l$ has intensity $u 1_{\{\tilde{D}'_{r+1} < \infty\}} \kappa_{e_{A', \tilde{B}}}$)

$$\begin{aligned} Q \otimes Q'(\mathcal{I}_1 \subset (\mathcal{I}(\mu) - x) \cap A \subset \mathcal{I}_1 \cup \mathcal{I}_2) &\stackrel{(6.13), (6.19), (6.21)}{\geq} \\ Q'(\bar{\mu}^+ = 0 \text{ and } \bar{\mu}^- = 0) Q(\sum_{l>r} \mu_l = 0) &= \\ \exp(- (u_+ P_{e_A}^{\mathbb{Z}^{d+1}}(\tilde{D}_{r+1} < \infty) + u P_{e_{A', \tilde{B}}}(\tilde{D}'_{r+1} < \infty))) &\geq \\ 1 - cu_+ \left\{ P_{e_A}^{\mathbb{Z}^{d+1}}(\tilde{D}_{r+1} < \infty) + P_{e_{A', \tilde{B}}}(\tilde{D}'_{r+1} < T_{\tilde{B}}) \right\} &\geq \\ 1 - cu_+ \left\{ \left(\sup_{z \in \partial_e B} P_z^{\mathbb{Z}^{d+1}}(H_A < \infty) \right)^r + \left(\sup_{z \in \partial_e B'} P_z(H_{A'} < T_{\tilde{B}}) \right)^r \right\} &\quad (6.23) \end{aligned}$$

To bound the last line of the above formula we will need:

Lemma 6.6. ($N \geq c(\varepsilon)$, $x \in \mathbb{T}_N \times [-\frac{N}{2}, \frac{N}{2}]$)

$$\sup_{z \in \partial_e B} P_z^{\mathbb{Z}^{d+1}}(H_A < \infty) \leq N^{-c_6(\varepsilon)}, \quad (6.24)$$

$$\sup_{z \in \partial_e B'} P_z(H_{A'} < T_{\tilde{B}}) \leq N^{-c_6(\varepsilon)}. \quad (6.25)$$

Proof. To prove (6.25) note that by the strong Markov property $\sup_{z \in \partial_e B'} P_z(H_{A'} < T_{\tilde{B}}) \leq \sup_{z \in \partial_e B'} P_z(H_{A'} < T_{C'}) + \sup_{z \in \partial_e C'} P_z(H_{\partial_e B'} < T_{\tilde{B}}) \times \sup_{z \in \partial_e B'} P_z(H_{A'} < T_{\tilde{B}})$ which implies

$$\sup_{z \in \partial_e B'} P_z(H_{A'} < T_{\tilde{B}}) \leq \frac{\sup_{z \in \partial_e B'} P_z(H_{A'} < T_{C'})}{\inf_{z \in \partial_e C'} P_z(T_{\tilde{B}} < H_{\partial_e B'})}. \quad (6.26)$$

By the invariance principle $\inf_{z \in \partial_e C'} P_z(T_{\mathbb{T}_N \times \{-N, N\}} < H_{B'}) \geq c$ for $N \geq c$ and by a one dimensional random walk estimate we see $\inf_{z \in \mathbb{T}_N \times \{-N, N\}} P_z(T_{\tilde{B}} < H_{B'}) \geq c \frac{1}{(\log N)^2}$, so $\inf_{z \in \partial_e C'} P_z(T_{\tilde{B}} < H_{\partial_e B'}) \geq c \frac{1}{(\log N)^2}$. We also have

$$\sup_{z \in \partial_e B'} P_z(H_{A'} < T_{C'}) \leq \sup_{z \in \partial_e B} P_z^{\mathbb{Z}^{d+1}}(H_A < \infty).$$

But Proposition 1.5.10, p. 36 of [12] implies that $\sup_{z \in \partial_e B} P_z^{\mathbb{Z}^{d+1}}(H_A < \infty) \leq N^{-2c_6(\varepsilon)}$, thus proving (6.24) and also, via (6.26), completing the proof of (6.25). \square

Remark 6.7. Let us record for later that a similar argument (introducing a set $C'' = B(x, \frac{N}{3}) \supset C'$ which plays the role of C') also proves

$$\sup_{z \in \partial_e C} P_z^{\mathbb{Z}^{d+1}}(H_{\partial_e B} < \infty) \leq N^{-c(\varepsilon)} \text{ and } \sup_{z \in \partial_e C'} P_z(H_{\partial_e B'} < T_{\tilde{B}}) \leq N^{-c(\varepsilon)}. \quad (6.27)$$

□

We now continue with the proof of Proposition 6.1. By (6.24) and (6.25) we see that

$$\begin{aligned} \left(\sup_{z \in \partial_e B} P_z^{\mathbb{Z}^{d+1}}(H_A < \infty) \right)^r + \left(\sup_{z \in \partial_e B'} P_z(H_{A'} < \infty) \right)^r &\leq c(cN^{-c_6})^r \\ &\stackrel{(6.15)}{\leq} cN^{-10(d+1)}. \end{aligned}$$

Thus (6.3) follows from (6.23). This completes the proof of Proposition 6.1. □

It remains to prove Proposition 6.2.

Proof of Proposition 6.2. For $w \in \mathcal{T}_B$ (see under (1.1) for the notation) let w^s denote the vertex at which w starts and let w^e denote the vertex at which it ends (i.e. stays constant). Note that for all $\mathbf{w} = (w_1, \dots, w_l) \in (\mathcal{T}_B)^{\times l}$

$$\begin{aligned} \xi_{\mathbb{Z}^{d+1}}^l(\mathbf{w}) &\stackrel{(6.9), (6.7)}{=} P_{e_A}^{\mathbb{Z}^{d+1}}(\tilde{D}_l < \infty = \tilde{R}_{l+1}, X_{(\tilde{R}_k + \cdot) \wedge \tilde{D}_k} = w_k, 1 \leq k \leq l) \\ &= e_A(w_1^s) \left(\prod_{i=1}^l r(w_i) \right) \left(\prod_{i=1}^{l-1} s_{\mathbb{Z}^{d+1}}(w_e^i, w_s^{i+1}) \right) t_{\mathbb{Z}^{d+1}}(w_l^e), \end{aligned} \quad (6.28)$$

where the last equality follows by several applications of the strong Markov property and where we define

$$\begin{aligned} r(w) &= P_{w^s}^{\mathbb{Z}^{d+1}}(X_{\cdot \wedge T_B} = w) = P_{w^s+x}(X_{\cdot \wedge T_{B'}} = w+x) \text{ for } w \in \mathcal{T}_B, \\ s_{\mathbb{Z}^{d+1}}(z, y) &= P_z^{\mathbb{Z}^{d+1}}(H_A < \infty, X_{H_A} = y) \text{ for } z \in \partial_e B, y \in \partial_i A \text{ and} \end{aligned} \quad (6.29)$$

$$t_{\mathbb{Z}^{d+1}}(z) = P_z^{\mathbb{Z}^{d+1}}(H_A = \infty) \text{ for } z \in \partial_e B. \quad (6.30)$$

Similarly for all $\mathbf{w} = (w_1, \dots, w_l) \in (\mathcal{T}_B)^{\times l}$

$$\begin{aligned} \xi_E^l(\mathbf{w}) &\stackrel{(6.8), (6.7)}{=} P_{e_{A', \tilde{B}}}(\tilde{D}'_l < T_{\tilde{B}} < \tilde{R}'_{l+1}, X_{(\tilde{R}'_k + \cdot) \wedge \tilde{D}'_k} - x = w_k, 1 \leq k \leq l) \\ &= e_{A', \tilde{B}}(w_1^s + x) \left(\prod_{i=1}^l r(w_i) \right) \left(\prod_{i=1}^l s_E(w_e^i, w_s^{i+1}) \right) t_E(w_l^e), \end{aligned} \quad (6.31)$$

where

$$s_E(z, y) = P_{z+x}(H_{A'} < T_{\tilde{B}}, X_{H_{A'}} = y+x) \text{ for } z \in \partial_e B, y \in \partial_i A, \quad (6.32)$$

$$t_E(z) = P_{z+x}(H_{A'} > T_{\tilde{B}}) \text{ for } z \in \partial_e B. \quad (6.33)$$

We will make a factor by factor comparison of the right-hand sides of (6.28) and (6.31) to obtain (6.10). For this we will need the following lemmas:

Lemma 6.8. $(N \geq c(\varepsilon), x \in \mathbb{T}_N \times [-\frac{N}{2}, \frac{N}{2}])$ For all $z \in \partial_i \mathbf{A}$

$$e_{\mathbf{A}}(z)(1 - cN^{-c(\varepsilon)}) \leq e_{\mathbf{A}', \tilde{B}}(z + x) \leq e_{\mathbf{A}}(z)(1 + cN^{-c(\varepsilon)}). \quad (6.34)$$

Lemma 6.9. $(N \geq c(\varepsilon), x \in \mathbb{T}_N \times [-\frac{N}{2}, \frac{N}{2}])$ For all $z \in \partial_e \mathbf{B}$

$$(1 - cN^{-c(\varepsilon)})t_{\mathbb{Z}^{d+1}}(z) \leq t_E(z) \leq (1 + cN^{-c(\varepsilon)})t_{\mathbb{Z}^{d+1}}(z). \quad (6.35)$$

Lemma 6.10. $(N \geq c(\varepsilon), x \in \mathbb{T}_N \times [-\frac{N}{2}, \frac{N}{2}])$ For all $z \in \partial_e \mathbf{B}$ and $y \in \partial_i \mathbf{A}$

$$(1 - cN^{-c(\varepsilon)})s_{\mathbb{Z}^{d+1}}(z, y) \leq s_E(z, y) \leq (1 + cN^{-c(\varepsilon)})s_{\mathbb{Z}^{d+1}}(z, y). \quad (6.36)$$

Before proving these lemmas we note that by comparing (6.28) and (6.31) and applying (6.34), (6.35) and (6.36) we get (6.10). The proof of Proposition 6.2 is thus done once we have proved Lemmas 6.8, 6.9 and 6.10. We start with Lemma 6.8:

Proof of Lemma 6.8. The upper bound follows by the argument in the proof of Lemma 4.4 of [20] (that lemma proves the upper bound with $B(0, 2[\frac{N^{1-\varepsilon}}{8}])$ in the place of \mathbf{A}' , but the special form of the radius and that the centre is at 0 plays essentially no role in the argument). The lower bound follows by the argument leading up to (6.4) of [21] (that formula is the upper bound in the case $x = 0$, but similarly the fact that $x = 0$ plays no essential role in the argument). \square

We now continue with the proof of Proposition 6.2 by proving Lemma 6.9.

Proof of Lemma 6.9. We will compare $t_E(z)$ and $t_{\mathbb{Z}^{d+1}}(z)$ with

$$t_{\mathbf{C}}(z) = P_z^{\mathbb{Z}^{d+1}}(H_{\mathbf{A}} > T_{\mathbf{C}}) \stackrel{(6.4), (6.5)}{=} P_{z+x}(H_{\mathbf{A}'} > T_{\mathbf{C}'}) \text{ for } z \in \partial_e \mathbf{B}. \quad (6.37)$$

It is obvious from (6.30) that $t_{\mathbb{Z}^{d+1}}(z) \leq t_{\mathbf{C}}(z)$, so to show the first inequality of (6.35) it suffices to show $(1 - cN^{-c(\varepsilon)})t_{\mathbf{C}}(z) \leq t_E(z)$. But this follows by the following upper bound on $t_{\mathbf{C}}(z)$:

$$\begin{aligned} t_{\mathbf{C}}(z) &\stackrel{(6.37)}{=} P_{z+x}(H_{\mathbf{A}'} > T_{\tilde{B}}) + P_z(T_{\tilde{B}} > H_{\mathbf{A}'} > T_{\mathbf{C}'}) \\ &\stackrel{(6.33), (6.37)}{\leq} t_E(a) + t_{\mathbf{C}}(z) \sup_{z' \in \partial_e \mathbf{C}'} P_{z'}(H_{\mathbf{A}'} < T_{\tilde{B}}) \\ &\stackrel{(6.25), \mathbf{B}' \subset \mathbf{C}'}{\leq} t_E(z) + t_{\mathbf{C}}(z)N^{-c(\varepsilon)}. \end{aligned}$$

To show the second inequality of (6.35) note that from (6.33), (6.37) and $\mathbf{C} \subset \tilde{B}$ it is obvious that $t_E(z) \leq t_{\mathbf{C}}(z)$, so it suffices to show $t_E(z) \leq (1 + cN^{-c(\varepsilon)})t_{\mathbb{Z}^{d+1}}(z)$. But this follows from by the following upper bound on $t_{\mathbf{C}}(z)$:

$$\begin{aligned} t_{\mathbf{C}}(z) &\stackrel{(6.37)}{=} P_z^{\mathbb{Z}^{d+1}}(H_{\mathbf{A}} = \infty) + P_z^{\mathbb{Z}^{d+1}}(\infty > H_{\mathbf{A}} > T_{\mathbf{C}}) \\ &\stackrel{(6.30), (6.37)}{\leq} t_{\mathbb{Z}^{d+1}}(z) + t_{\mathbf{C}}(z) \sup_{z' \in \partial_e \mathbf{C}} P_{z'}^{\mathbb{Z}^{d+1}}(H_{\mathbf{A}} < \infty) \\ &\stackrel{(6.24), \mathbf{B} \subset \mathbf{C}}{\leq} t_{\mathbb{Z}^{d+1}}(z) + t_{\mathbf{C}}(z)cN^{-c(\varepsilon)}. \end{aligned}$$

This completes the proof of Lemma 6.9. \square

Finally we prove Lemma 6.10; the argument is close to that of Lemma 5.3 in [21] and Lemma 3.2 of [20].

Proof of Lemma 6.10. For $z \in \partial_e \mathbf{B}$, $y \in \partial_i \mathbf{A}$ we will compare $s_{\mathbb{Z}^{d+1}}(z, y)$ and $s_E(z, y)$ with

$$s_C(z, y) = P_z^{\mathbb{Z}^{d+1}}(H_A < T_C, X_{H_A} = y) \stackrel{(6.4), (6.5)}{=} P_{z+x}(H_{A'} < T_{C'}, X_{H_{A'}} = y + x). \quad (6.38)$$

Recalling (6.29) and using the decomposition $W = \{T_C < H_A\} \cup \{H_A < T_C\}$, and similarly recalling (6.32) and using the decomposition $\mathcal{T}_{\tilde{B}} = \{T_{C'} < H_{A'}\} \cup \{H_{A'} < T_{C'}\}$, we see that

$$\begin{aligned} s_C(z, y) &\leq s_{\mathbb{Z}^{d+1}}(z, y) = s_C(z, y) + P_z^{\mathbb{Z}^{d+1}}(T_C < H_A < \infty, X_{H_A} = y) \text{ and} \\ s_C(z, y) &\leq s_E(z, y) = s_C(z, y) + P_{z+x}(T_{C'} < H_{A'} < T_{\tilde{B}}, X_{H_{A'}} = y). \end{aligned} \quad (6.39)$$

To prove (6.36) it suffices to show

$$P_z^{\mathbb{Z}^{d+1}}(T_C < H_A < \infty, X_{H_A} = y) \leq cN^{-c(\varepsilon)} s_{\mathbb{Z}^{d+1}}(z, y) \text{ and} \quad (6.40)$$

$$P_{z+x}(T_{C'} < H_{A'} < T_{\tilde{B}}, X_{H_{A'}} = y) \leq cN^{-c(\varepsilon)} s_E(z, y), \quad (6.41)$$

since then $s_{\mathbb{Z}^{d+1}}(z, y)(1 - cN^{-c(\varepsilon)}) \leq s_E(z, y)$ by using the upper right-hand side of (6.39), (6.40) and then the lower left-hand side of (6.39), and similarly $s_E(z, y)(1 - cN^{-c(\varepsilon)}) \leq s_{\mathbb{Z}^{d+1}}(z, y)$.

We start with (6.40). We have

$$\begin{aligned} &\sup_{z \in \partial_e \mathbf{B}} P_z^{\mathbb{Z}^{d+1}}(T_C < H_A < \infty, X_{H_A} = y) \\ &\leq \sup_{z' \in \partial_e \mathbf{C}} P_{z'}^{\mathbb{Z}^{d+1}}(H_{\partial_e \mathbf{B}} < \infty) \sup_{z'' \in \partial_e \mathbf{B}} P_{z''}^{\mathbb{Z}^{d+1}}(H_A < \infty, X_{H_A} = y) \\ &\stackrel{(6.27), (6.29)}{\leq} cN^{-c(\varepsilon)} \sup_{z'' \in \partial_e \mathbf{B}} s_{\mathbb{Z}^{d+1}}(z'', y). \end{aligned} \quad (6.42)$$

Note that the map $z \rightarrow P_z^{\mathbb{Z}^{d+1}}(H_A < \infty, X_{H_A} = y)$ is positive harmonic on $\mathbb{Z}^{d+1} \setminus \mathbf{A}$ so that by Harnack's inequality (Theorem 1.7.2 p. 42 of [12]) and a standard covering argument we get $\sup_{z'' \in \partial_e \mathbf{B}} s_{\mathbb{Z}^{d+1}}(z'', y) \leq c \inf_{z'' \in \partial_e \mathbf{B}} s_{\mathbb{Z}^{d+1}}(z'', y)$. Combining this with (6.42) we get (6.40).

It remains to show (6.41). Similarly to (6.42) we have:

$$\sup_{z \in \partial_e \mathbf{B}} P_{z+x}(T_{C'} < H_{A'} < T_{\tilde{B}}, X_{H_{A'}} = y) \stackrel{(6.27), (6.32)}{\leq} cN^{-c(\varepsilon)} \sup_{z \in \partial_e \mathbf{B}} s_E(z, y). \quad (6.43)$$

Now the map $z \rightarrow P_z(H_{A'} < T_{\tilde{B}}, X_{H_{A'}} = y)$ is positive harmonic on $\tilde{B} \setminus \mathbf{A}' \supset \mathbf{C}' \setminus \mathbf{A}'$. Since $\mathbf{C}' \setminus \mathbf{A}'$ can be identified as a subset of \mathbb{Z}^{d+1} we have similarly to above by Harnack's inequality that $\sup_{z \in \partial_e \mathbf{B}} s_E(z, y) \leq c \inf_{z \in \partial_e \mathbf{B}} s_E(z, y)$. Combining this with (6.43) we get (6.41). This completes the proof of Lemma 6.10. \square

This also completes the proof of Proposition 6.2. \square

All the steps in the proof of Theorem 0.1 and its corollaries have now been completed. We conclude with an open question and a comment on the use of Theorem 4.1 as a “transfer mechanism”.

Remark 6.11. (1) The Gumbel distribution has been proven to arise as a distributional limit for rescaled cover times of certain finite graphs (see [11, 13]). One important graph in the study of cover times for which a Gumbel distributional limit has been conjectured (see Chapter 7, Section 2.2, p. 23 of [3]), but not proved, is the discrete torus $\mathbb{T}_N = (\mathbb{Z}/N\mathbb{Z})^d, d \geq 3$. It is an open question whether the methods of the proof of Theorem 0.1 could be used to prove that conjecture. A strategy could be to reduce it to the statement (1.45) with the help of a coupling with random interlacements. For bounded u and fixed δ a coupling of random interlacements and the trace of random walk in the torus (in *one* local box) has been produced in [23].

(2) A coupling of random walk with random interlacements can be used as a “transfer mechanism” to reduce the proofs of properties of random walk in the cylinder to proofs of properties purely in term of random interlacements (as we reduced Theorem 0.1 to (1.45)). Sometimes such transfers require the use of both inclusions (cf. (4.2)) simultaneously and therefore need a coupling of random walk with *joint* random interlacements, such as Theorem 4.1. An example arises when using the random interlacement concept of strong supercriticality of levels $u > 0$ (see Definition 2.4 of [23]) to “patch up” components of the vacant set $(X(0, n))^c, n \geq 1$, in various local boxes where the walk is coupled with random interlacements (as was done in the case of the torus in Proposition 2.7, see also Lemma 2.6, of [23]). For instance if one could prove that all $u < u^*$ are strongly supercritical (where u^* is the critical parameter of interlacement percolation, see (0.13) of [22] and Remark 2.5 (2) of [23]) then Theorem 4.1 would be the kind of coupling that could be used to derive from this, using the aforementioned “patching”, the “correct” lower bound on the disconnection time T_N of the cylinder, (and thus improve on Theorem 7.3 of [21], see also Remark 7.5 (2) of [21]).

□

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